Realization

Given

$$\dot{x} = Ax + Bu \qquad , \quad x \in \mathbb{R}^n$$
$$y = Cx + Du$$

we have

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}Bu(s)\,ds$$

Setting $x(0) = 0 \Rightarrow$

$$y(t) = \int_0^t Ce^{A(t-s)}Bu(s)\,ds + Du$$

Recall the input-output description:

$$y(t) = \int_0^t G(t-s)u(s)\,ds + Du$$

$$\Rightarrow \qquad G(t-s) = Ce^{A(t-s)}B$$
$$\text{or} \qquad G(t) = Ce^{At}B \qquad (*)$$

State space $\Rightarrow$ input-output

Realization: given $G(t)$, can we find (A, B, C) s.t. $(*)$ holds

Definition: (A, B, C) that makes $(*)$ hold is called a realization of $G(t-s)$, and the dimension of the realization is the order of A.

Definition: We say (A, B, C) is a minimal realization if no other realization has lower dimension.

Given $G(t-s)$, if $\exists\ (A, B, C)$ s.t.

$$Ce^{At}B = G(t)$$

$$\Rightarrow \quad \mathcal{L}(Ce^{At}B) = \mathcal{L}(G(t)) := R(s)$$

where $R(s)$ is called transfer matrix of the system

and $\mathcal{L}(Ce^{At}B) = C\mathcal{L}(e^{At})B = C(sI-A)^{-1}B$

$\Rightarrow \quad C(sI-A)^{-1}B = R(s)$ if (A, B, C) is a realization.

Recall that

$$(sI-A)^{-1} = \frac{1}{\det(sI-A)} \operatorname{adj}(sI-A)$$

where $\det(sI-A)$ is a polynomial of s of degree n, and each element in $\operatorname{adj}(sI-A)$ is a polynomial of s of degree highest $n-1$.

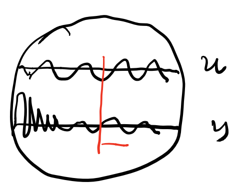
$\Rightarrow$ $C(sI-A)^{-1}B$ is a rational matrix function of s that is strictly proper.

$\Rightarrow$ a necessary condition for a given $R(s)$ to be realisable is that it is strictly proper.

How do measure $R(s)$ ($G(t)$) for a given input-output system?

We assume the steady-state response exists!

take input as $u(t) = \sin \omega t$, $\omega = \omega_1, \omega_2, \ldots$.



$$y(t) = \int_{t_0}^{t} G(t-s)\,u(s)\,ds$$

Steady-state response: $y(t) = \int_{-\infty}^{t} G(t-s)\,u(s)\,ds$

$$u(t) = \sin \omega t = \operatorname{Im} e^{j\omega t} = \operatorname{Im}(\cos\omega t + j \sin \omega t)$$

$$\Rightarrow \quad y(t) = \operatorname{Im} \int_{-\infty}^{t} G(t-s)\, e^{j\omega s}\, ds$$

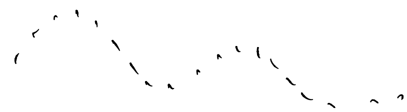
$$\overset{r=t-s}{=} \operatorname{Im} \int_{0}^{\infty} G(r)\, e^{j\omega(t-r)}\, dr$$

$$= \operatorname{Im} \int_{0}^{\infty} G(r)\, e^{-j\omega r}\, dr \; e^{j\omega t}$$

$$= \operatorname{Im} R(j\omega)\, e^{j\omega t}$$

$\Rightarrow$ the magnitude of $y(t)$ is $|R(j\omega)|$

and the phase is $\arg R(j\omega)$.



We show after the break being strictly proper for $R(s)$ is sufficient for state space realization.

Comment: if a given $R(s)$ is only proper,

then we let $\bar{R}(s) = R(s) - R(\infty)$

$\Rightarrow$ $\bar{R}(s)$ is strictly proper, and

we can see $D = R(\infty)$.

Sufficiency:

Let $R(s)$ be a $m \times k$ strictly proper transfer matrix, i.e. $y(s) = R(s)u(s)$

1. Standard reachable realisation:

Let $\chi(s)$ be the least common denominator of all the elements of $R(s)$, let $r = \deg \chi(s)$

$$\Rightarrow \quad \chi(s) = s^r + a_1 s^{r-1} + \cdots + a_r$$

$$\Rightarrow \quad \chi(s)R(s) = N_0 + N_1 s + \cdots + N_{r-1} s^{r-1} := N(s)$$

Example: $R(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ \frac{-1}{(s+1)(s+2)} & \frac{1}{s+2} \end{bmatrix}$

$$\Rightarrow \quad \chi(s) = (s+1)(s+2) = s^2 + 3s + 2$$

$$N(s) = \chi(s)R(s) = \begin{bmatrix} s+2 & s+1 \\ -1 & s+1 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}}_{N_0} + \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{N_1} s$$

Then

$$A = \begin{bmatrix} 0 & I_k & & \\ \vdots & & \ddots & \\ 0 & \cdots & & I_k \\ -a_r I_k & \cdots & & -a_1 I_k \end{bmatrix}_{rk \times rk}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_k \end{bmatrix}_{rk \times k}$$

$$C = [N_0 \ \cdots \ N_{r-1}]_{m \times rk}$$

gives a realization of $R(s)$, i.e.

$$C(sI-A)^{-1}B = R(s)$$

Example: $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & -3 & 0 \\ 0 & -2 & 0 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$C = \begin{bmatrix} 2 & 1 & 1 & 2 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

Proof: Let $(sI-A)\mathcal{X}(s) = B$

$$\Rightarrow \quad \mathcal{X}(s) = (sI-A)^{-1}B,$$

where $\mathcal{X}(s) = \begin{bmatrix} X_1(s) \\ \vdots \\ X_r(s) \end{bmatrix}$

$$\Rightarrow \quad (sI-A) = \begin{bmatrix} sI_k & -I_k & 0 & \cdots & 0 \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & -I_k \\ a_rI_k & \cdots & & & sI_k + a_1I_k \end{bmatrix} \begin{bmatrix} X_1(s) \\ \vdots \\ X_r(s) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_k \end{bmatrix}$$

$$\Rightarrow \begin{cases} sX_1(s) = X_2(s) \\ sX_i(s) = X_{i+1}(s), \quad i = 1, \cdots, r-1 \\ a_rX_1 + \cdots + a_2X_{r-1} + sX_r + a_1X_r = I_k \end{cases}$$

$$\Rightarrow \quad X_i(s) = s^{i-1}X_1(s)$$

$$\Rightarrow \quad X_i(s) = \frac{s^{i-1}}{\chi(s)} I_k$$

$$\Rightarrow \quad C(sI-A)^{-1}B = C\mathcal{X}(s) = \frac{1}{\chi(s)} C \underbrace{\begin{bmatrix} I_k \\ \vdots \\ s^{r-1}I_k \end{bmatrix}}_{N} = R(s)$$

2. Standard observable realization.

Since $R(\infty) = 0 \Rightarrow R(s)$ is analytic at $s = \infty$. $\Rightarrow \exists$ Laurant expansion.

$$R(s) = R_1 s^{-1} + R_2 s^{-2} + \cdots + R_k s^{-k} + \cdots$$

on the other hand, if (A, B, C) is realization, then

$$C(sI-A)^{-1}B = R(s)$$

Since $\quad C(sI-A)^{-1}B = CBs^{-1} + CABs^{-2} + \cdots$

$$\Rightarrow \qquad CA^{i-1}B = R_i \quad, i=1,\cdots,\infty$$

R_i's are called the markov parameters.

Since $\quad X(s)R(s) = N(s)$ — a polynomial matrix of s

$$\Rightarrow \quad (s^r + a_1 s^{r-1} + \cdots + a_r)(R_1 s^{-1} + R_2 s^{-2} + \cdots) = N_0 + \cdots + N_{r-1}s^{r-1}$$

$$\Rightarrow \quad \boxed{R_{r+i} = -a_1 R_{r+i-1} - \cdots - a_r R_i \quad, i=1,\cdots,}$$

then

$$A = \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & \ddots & I_m \\ -a_r I_m & & \cdots & & -a_1 I_m \end{bmatrix}, \quad B = \begin{bmatrix} R_1 \\ \vdots \\ R_r \end{bmatrix}$$

$$C = [I_m \; 0 \cdots 0]$$

we just need to $\quad CA^{i-1}B = R_i \,, i=1,\cdots,\infty$