Kalman filter

Consider

$$x(t+1) = A(t)x(t) + B(t)v(t)$$

$$y(t) = C(t)x(t) + D(t)w(t)$$

where

$$E\{v(t)v^T(s)\} = Q\,\delta(t-s), \quad E\{w(t)w^T(s)\} = R\,\delta(t-s)$$

$$E\{v(t)w^T(s)\} = 0$$

Denote $H_t(y) = \text{span}\{y_1(0), \cdots, y_m(0), \cdots, y_1(t), \cdots, y_m(t)\}$

$$\hat{x}(t) = E^{H_{t-1}(y)} x(t)$$

$$\hat{x}_t(t) = E^{H_t(y)} x(t)$$

$$\tilde{x}(t) = x(t) - \hat{x}(t)$$

$$P(t) = E\{\tilde{x}(t)\tilde{x}(t)^T\}$$

We have derived: (by orthogonal projection)

Measurement update: $\hat{x}_t(t) = \hat{x}(t) + K(y(t) - C(t)\hat{x}(t))$

where $K = P(t)C^T(t)(C(t)P(t)C^T(t) + D(t)RD^T(t))^{-1}$

— Kalman gain

Time update:

$$\hat{x}(t+1) = A\,\hat{x}_t(t) \quad \left(\hat{x}(t+1) = E^{-H_t(y)} x(t+1)\right)$$

$$\Rightarrow \quad \hat{x}(t+1) = A(t)\,\hat{x}(t) + AK\left(y(t) - C(t)\,\hat{x}(t)\right)$$

Now let $P_t(t) = E\left\{(x(t) - \hat{x}_t(t))(x(t) - \hat{x}_t(t))^T\right\}$

$$\Rightarrow \quad P_t(t) = P(t) - P(t)C^T(CP(t)C^T + DRD^T)^{-1}CP(t)$$

Denote $P(t+1) = E\left\{(x(t+1) - \hat{x}(t+1))(x(t+1) - \hat{x}(t+1))^T\right\}$

$$\tilde{x}(t+1) = x(t+1) - \hat{x}(t+1)$$

$\Rightarrow$

$$\tilde{x}(t+1) = A\,\tilde{x}(t) - AKC\,\tilde{x}(t) - AKDW + BV$$

$$\begin{cases} P(t+1) = (A - AKC)P(t)(A - AKC)^T + \\ \qquad AKDRD^TK^TA^T + BQB^T \\ P(0) = P_0 \end{cases}$$

Since our measurement starts at $t=0$

$$\hat{x}(0) = E^{-H_{-1}(y)} x(0) = 0.$$

$$\Rightarrow \quad P_0 = E\left\{x(0)\,x^T(0)\right\}$$

Kalman filter for continuous time systems

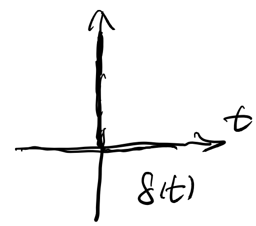
Consider

$$\dot{x}(t) = A(t)x(t) + B(t)v(t) \quad , \; x \in \mathbb{R}^n$$
$$y(t) = C(t)x(t) + D(t)w(t) \qquad y \in \mathbb{R}^m$$
$$x(0) = x_0$$

Where x_0, $v(t)$, and $w(t)$ are uncorrelated.

$$E\left\{ \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} \begin{bmatrix} v(s) \\ w(s) \end{bmatrix}^T \right\} = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \delta(t-s)$$

Where $\int_{-\infty}^{\infty} f(t)\delta(t)\,dt = f(0)$



Bottom line: find the "best" estimator $\hat{x}(t)$ for $x(t)$ based on $y(t)$.

Recall
$$\dot{\hat{x}}(t) = Ax + L(y(t) - C\hat{x})$$
$$= (A - LC)\hat{x}(t) + Ly(t)$$

an observer can be viewed as a linear system with $y(t)$ as the input.

$$\Rightarrow \quad \int_0^t G(t,s)\,y(s)\,ds$$

Duality between control and observation

Define

$$\dot{z} = -A^T(t) z + C^T(t) u$$

$$z(T) = a$$

$$\frac{d}{dt}(z^T(t) x(t)) = z^T B v(t) - u^T D w(t) + u^T y$$

$$a^T x(T) - z(0)^T x(0) = \int_0^T (\qquad) dt$$

$$a^T x(T) - \int_0^T u^T y \, dt = z^T(0) x(0) + \int_0^T (z^T B v - u^T D w) \, dt$$

$$E\{(a^T x(T) - \int_0^T u^T y \, dt)^2\}$$

$$= z(0)^T P_0 z(0) + \int_0^T (z^T \tilde{Q} z + u^T \tilde{R} u) \, dt$$

where, $\tilde{Q} = B Q B^T$, $\tilde{R} = D R D^T(t)$

$\Rightarrow$ Find u s.t.

$$\begin{cases} \min \; z(0)^T P_0 z(0) + \int_0^T (z^T \tilde{Q} z + u^T \tilde{R} u) \, dt \\ \dot{z} = -A^T z + C^T u \\ z(T) = a \end{cases}$$

Recall that for

$$\min \quad x(T)^T S x(T) + \int_0^T (x^T Q x + u^T R u)\, dt$$

$$\dot{x} = Ax + Bu$$

$$x(0) = x_0$$

$$u^* = -R^{-1} B^T P(t) x$$

where $\dot{P} = -A^T P - PA + PBR^{-1}B^T P - Q$

$$P(T) = S$$

we can let $\bar{t} = T - t$

$$\boxed{K = P(t) C^T(t) \tilde{R}^{-1}}$$

$$\Rightarrow \quad u^* = \tilde{R}^{-1} C P(t) z(t)$$

$$= (PC^T \tilde{R}^{-1})^T z(t) = K^T z$$

$$\begin{cases} \dot{P} = AP + PA^T - PC^T \tilde{R}^{-1} C P + \tilde{Q} \\ P(0) = P_0 \end{cases}$$

we plugin u^*:

$$\dot{z} = -A^T z + C^T (PC^T \tilde{R}^{-1})^T z$$

$$= -(A - PC^T \tilde{R}^{-1} C)^T z$$

$$= -(A - KC)^T z$$

$$z(T) = a$$

Recall if $\Phi(t,s)$ is the state transition matrix for $A(t)$, then $\Phi^T(s,t)$ is the state transition matrix for $-A^T(t)$

$$\left(\frac{\partial \Phi(t,s)}{\partial s} = -\Phi(t,s)A(s) \right)$$

Now let $\Phi(t,s)$ be the state transition matrix for $A(t)-K(t)C(t)$

$$z(t) = \Phi^T(0,t)z(0)$$

$$\Rightarrow z(T) = \Phi^T(0,T)z(0)$$

$$z(0) = (\Phi^T(0,T))^{-1}z(T) = \Phi^T(T,0)z(T)$$

$$\Rightarrow z(t) = (\Phi(T,0)\Phi(0,t))^T z(T)$$

$$= \Phi^T(T,t)z(T)$$

$$= \Phi^T(T,t)a$$

$$\Rightarrow \boxed{u^* = K^T(t)\Phi^T(T,t)a}$$

(Some writing got erased accidentally)

$$\Rightarrow \quad \int_0^t \Phi(t,s)\, K(s)\, y(s)\, ds$$ gives the best estimator.

$$\Rightarrow \quad \dot{\hat{x}}(t) = (A(t) - K(t)C(t)\, \hat{x}(t) + K(t)\, y(t))$$

$$= A(t)\, \hat{x}(t) + K(t)(y(t) - C(t)\, \hat{x}(t))$$

$$\hat{x}(0) = 0$$