LQ and ARE

Consider

$$\min J(u) = \int_0^{t_1} (x^T Q x + u^T R u)\, dt$$

s.t. $\dot{x} = Ax + Bu$

$x(0) = x_0$

let $t_1 \to \infty \Rightarrow J(u) = \int_0^{\infty} (x^T Q x + u^T R u)\, dt$

under what conditions this is well posed?

- Assume (A, B) reachable

$\Rightarrow \exists$ feasible $u(t) \in \mathcal{U}$ i.e.

$$\int_0^{\infty} (x^T Q x + u^T R u)\, dt < \infty \quad \forall u \in \mathcal{U}.$$

$x^T Q x = \|y\|^2 = \|Cx\|^2 = x^T C^T C x$

We have also shown that $P(t - t_1)$ for

$$\dot{P} = -A^T P - PA + PBR^{-1}B^T P - Q$$

$P(t_1) = 0$

$$\lim_{t_1 \to \infty} P(0 - t_1) := P_\infty \geq 0$$

If for a function $\lim_{t \to \infty} F(t) = F_\infty$ exists

what is $\lim_{t \to \infty} \dot{F}(t) = 0$?

This is true if $\dot{F}(t)$ is uniformly continuous

Counter example: $\frac{\sin(t^3)}{1+t}$

Since $P(t-t_1)$ is indeed uniformly continuous

$\Rightarrow \dot{P} \to 0 \Rightarrow P_\infty$ satisfies

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$ Algebraic Riccati equation (ARE)

Now under what conditions $P > 0$

a. If Q is positive definite, might be too strong

We decompose now $Q = C^T C$

Claim: if (C, A) is observable, then $P > 0$.

Since $V(x_0, t_1) = x_0^T P(0-t_1) x_0$ is the optimal cost for the time interval $[0, t_1]$ and $V(x_0, t_1)$ thus $P(0-t_1)$ is non-decreasing.

We only need to show for some $T > 0$,

$P(0-T) > 0$.

Suppose $P(0-T)$ is not positive definite, $\Rightarrow$

$\exists\ x_0 \neq 0$ s.t. $P(0-T)x_0 = 0$

$\Rightarrow V(x_0, T) = x_0^T P(0-T) x_0 = 0$

and $V(x_0, T) = \min \int_0^T (x^T C^T C x + u^T R u)\, dt = 0$

$\Rightarrow u^*(t) \equiv 0 \quad \forall t \in [0, T]$

and $\begin{cases} Cx(t) \equiv 0 & \forall t \in [0, T] \\ \dot{x} = Ax \end{cases}$

$\Rightarrow$

$$Cx(t) \equiv 0 \Rightarrow Cx^{(k)}(t) \equiv 0$$

$$\Rightarrow CA^k x(t) \equiv 0 \quad k = 0, \cdots, \infty$$

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x(t) \equiv 0$$

$\Rightarrow$ $x(t) \equiv 0$ if (C, A) is observable

$$\boxed{\text{Thus, } P > 0 \text{ if } (C, A) \text{ is observable}}$$

Claim: for any $u(t) \in \mathcal{U}$, $x(t) \to 0$ as $t \to \infty$. i.e. if

$$\int_0^\infty (x^T Q x + u^T R u)\, dt < \infty, \text{ then}$$

$$\boxed{\lim_{t \to \infty} x(t) = 0.}$$

$$\Rightarrow \int_T^{T+1} (x^T Q x + u^T R u)\, dt \quad T >> 1 \to 0.$$

Now we can discuss the optimal control for infinite time horizon.

Recall that for the case of finite time interval,

$$J(u) - x_0^T P(0 - t_1) x_0 = \int_0^{t_1} (u + R^{-1} B^T P x)^T R (u + R^{-1} B^T P x)\, dt$$

$$- x^T(t_1) P(t_1 - t_1) x(t_1)$$

provided P satisfies the DRE.

Now as $t_1 \to \infty$ P becomes the solution to ARE.

If $x(t_1) \to 0$ as $t_1 \to \infty$, then

$u^* = -R^{-1}B^T P x$ would be the optimal control, provide u^* is feasible.

$\Rightarrow$ we need to check if $x(t)$ to

$$\dot{x} = Ax - BR^{-1}B^T P x$$
$$= (A - BR^{-1}B^T P)x$$

converges to 0 as $t \to \infty$ or not

i.e. if $A - BR^{-1}B^T P$ is a stable matrix

Thm: $A - BR^{-1}B^T P$ is a stable matrix if $P > 0$, provided (A, B) is reachable and (C, A) is observable.

Proof: Assume $P > 0$, and let

$$V(x) = x^T P x$$

$$\dot{V} = \dot{x}^T P x + x^T P \dot{x}$$

$$= -x(t)^T \underbrace{(PBR^{-1}B^T P + C^T C)}_{\geq 0} x(t) \leq 0$$

$$\Rightarrow \quad V(x(t)) \leq V(x_0) \quad (\lambda_{m}\|x\|^2 \leq x^T P x \leq \lambda_{max}\|x\|^2)$$

$$\Rightarrow \quad \|x(t)\| \leq \gamma \|x_0\|$$

$\Rightarrow$ the closed-loop system is stable.

If there are eigenvalues on the imaginary axis, $\Rightarrow \exists$ periodic solution $x(t+T)=x(t)$

$$\Rightarrow \quad V(x(T)) = V(x_0))$$

$$\Rightarrow \quad \dot{V} \equiv 0 \quad \forall\, t\in[0,T]$$

$$\Rightarrow \quad x^T(t) PBR^{-1}B^T P x(t) \equiv 0 \quad \forall\, t\in[0,T]$$

$$x^T C^T C x(t) \equiv 0 \quad \forall\, t\in[0,T]$$

$$\Rightarrow \quad B^T P x(t) \equiv 0$$

$$C x(t) \equiv 0$$

$$\Rightarrow \quad \dot{x} = (A - BR^{-1}B^T P)x$$

$$= Ax$$

$$C x(t) \equiv 0 \Rightarrow C x^{(k)}(t) \equiv 0 \Rightarrow CA^k x(t) \equiv 0$$

$$\Rightarrow \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x(t) \equiv 0 \Rightarrow x(t) \equiv 0$$

$\Rightarrow$ $A - BR^{-1}B^T P$ is a stable matrix!

$\Rightarrow$ $u^* = -R^{-1}B^T P x$ is the optimal control.

Kalman filter

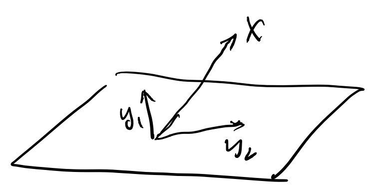
Orthogonal projection over Hilbert space

Example: suppose $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ is given

we want to use $y_1, \cdots, y_m \in \mathbb{R}^n$ to best estimate x, i.e. find

$$k_1 y_1 + \cdots + k_m y_m := \hat{x} \quad \text{s.t.}$$

$\|x - \hat{x}\|^2$ is minized.



denote $k = [k_1, \cdots k_m]$, $Y = [y_1, \cdots, y_m]^T$

$\Rightarrow \quad \hat{x}^T = kY$

$\Rightarrow$ Find k s.t. $\|kY - x^T\| \to$ minimal

Since $\|kY - x^T\|^2 = (kY - x^T)(kY - x^T)^T = \cdots$

$$\frac{\partial \|kY - x^T\|^2}{\partial k} = 0 .$$

$$\Rightarrow \quad k^* = x^T Y^T (Y Y^T)^{-1}$$

$$\Rightarrow \quad \hat{x}^T = k^* Y \quad \Rightarrow \hat{x}$$

now we let $x = x_1 e_1 + \cdots + x_n e_n$, where

$$e_i \cdot e_j = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

$$\Rightarrow \quad x \cdot y_i = x_1 y_1^i + \cdots + x_n y_n^i$$

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} \qquad [y] = \text{span}\{y_1, \cdots, y_m\}$$

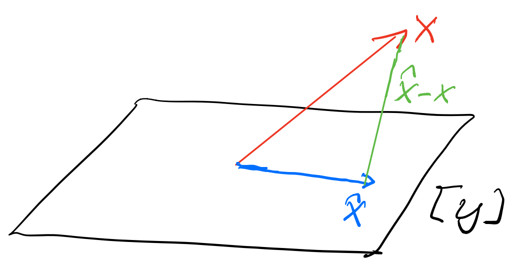
$$\Rightarrow \quad \hat{x} = k\, y \qquad m \in [y]$$

$$\Rightarrow \quad k^* = x \cdot y^T (y \cdot y^T)^{-1} \qquad \Rightarrow \quad m = a_1 y_1 + \cdots + a_m y_m$$

$$\Rightarrow \quad \hat{x} = \underbrace{x \cdot y^T (y \cdot y^T)^{-1}}_{\text{the optimal gain}} y \qquad = [y_1 \cdots y_m] \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} = y^T a$$

$$(\hat{x} - x) \cdot y^T a = (x \cdot y^T (y \cdot y^T)^{-1} y \cdot y^T a - x \cdot y^T a)$$

$$= 0.$$



$\Rightarrow \quad x - \hat{x}$ is orthogonal to $[y]$!