LQ optimal control

Consider

$$\min J(u) = \int_{t_0}^{t_1} \left(x^T(t) Q x(t) + u^T(t) R u(t) \right) dt + x^T(t_1) S x(t_1)$$

s.t. $\quad \dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n$

$\quad\quad x(t_0) = x_0 \qquad u \in \mathbb{R}^m$

Let $\quad V(x(t), t) = x^T(t) P(t) x(t)$

and $\quad$ assume $V(x(t_0), t_0) = x_0^T P(t_0) x_0$

is the optimal cost.

$$\frac{dV}{dt} = x^T(t) \left(A^T P(t) + P(t) A + \dot{P} \right) x(t) + u^T B^T P(t) x + x^T P B u$$

$$\Rightarrow \quad V(x(t_1), t_1) - V(x(t_0), t_0) = \int_{t_0}^{t_1} \frac{dV}{dt} = \cdots$$

$$\Rightarrow \quad J(u) - V(x_0, t_0) = \int_{t_0}^{t_1} \Big[x^T(t) \big(\underbrace{A^T P + PA - PBR^{-1}B^T P + Q + \dot{P}}_{0} \big) x(t)$$

$$+ \underbrace{\left(u(t) + R^{-1}B^T P(t) x(t) \right)^T R \left(u + R^{-1} B^T P x \right)}_{\geq 0} \Big] dt$$

$$+ x^T(t_1) \big(\underbrace{S - V(x(t_1), t_1)}_{0} \big) x(t_1)$$

$$\Rightarrow \begin{cases} \dot{P} = -A^T P - PA + PBR^{-1}B^T P - Q \\ P(t_1) = S \end{cases}$$

DRE - dynamical Riccati equation

$$\Rightarrow \quad J(u) - V(x_0, t_0) = \int_{t_0}^{t_1} (u + R^{-1}B^T P x)^T R (u + R^{-1}B^T P x)\, dt \geq 0$$

$$u^* = -R^{-1}B^T P(t) x(t) \Rightarrow J(u^*) = V(x_0, t_0) = x_0^T P(t_0) x_0$$

optimal !

Plugin u^* to the system,

$$\dot{x} = (\underbrace{A - BR^{-1}B^T P(t)}_{A_K(t)})x \quad \text{- closed-loop system}$$

$$\Rightarrow \quad x^*(t) = \Phi_K(t, t_0) x_0$$

$$\Rightarrow \quad u^* = -R^{-1}B^T P(t) \Phi_K(t, t_0) x_0 \quad \text{open-loop}$$

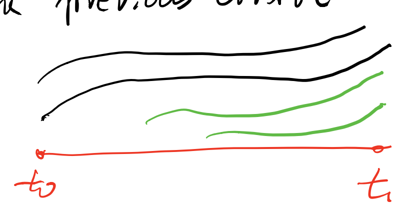
$(\Rightarrow$ if $t_0 = t$, $u^* = -R^{-1}B^T P(t) X(t))$

$\bar{t}_1 > t_1$

$$\int_{t_0}^{\bar{t}_1} (\ )\, dt = \int_{t_0}^{t_1} (\ )\, dt + \int_{t_1}^{\bar{t}_1} (\ )\, dt \geq \int_{t_0}^{t_1} (\ )\, dt \geq J(u^*)$$

Bellman's principle:

An optimal control has the property that no matter what the previous control has been, the remaining control must constitute an optimal control with respect to the state resulting from the previous control.



Solving DRE

we plug in u^* $\Rightarrow$

$$\hat{x} = (A - BR^{-1}B^T P(t))x \qquad (*)$$

Let $X_{n\times n}$ be a fundamental matrix for $(*)$
with $X(t_1) = I$, i.e.

$$\dot{X} = (A - BR^{-1}B^T P(t))X$$

$$= AX - BR^{-1}B^T Y(t) \qquad Y(t) := P(t)X(t)$$

$$\dot{Y} = \dot{P}X + P\dot{X} \qquad Y(t_1) = S$$

$$= -QX - A^T Y$$

$$\begin{bmatrix} \dot{X} \\ Y \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

$$\begin{bmatrix} X(t_1) \\ Y(t_1) \end{bmatrix} = \begin{bmatrix} I \\ S \end{bmatrix}$$

$\Rightarrow$ we can calculate $e^{\begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} t}$!

$\Rightarrow$ $P(t) = X^{-1}(t) Y(t)$

In general, given

$$\dot{x} = Ax$$
$$x(t_1) = x_1$$

we know $x(t) = e^{A(t-t_0)} x(t_0) \Rightarrow$

$$x_1 = e^{A(t_1-t_0)} x(t_0)$$

$$\Rightarrow \quad x(t_0) = e^{-A(t_1-t_0)} x_1$$

$$\Rightarrow \quad x(t) = e^{A(t-t_0)} e^{-A(t_1-t_0)} x_1$$
$$= e^{A(t-t_1)} x_1$$

$\Rightarrow$ $P(t)$ can be expressed as $P(t-t_1)$

$P(t_0) = P(t_0-t_1)$ while $P(t_1) = S \geq 0$.

$\underline{x^T P(t^1) x \geq x^T P(t^2) x}$ $\quad t_1 > t^2 > t^1 \geq t_0$

Fixed end-point problems

We consider

$$J(u) = \int_{t_0}^{t_1} [x^T(t) Q x(t) + u^T(t) R u(t)] \, dt$$

and $\quad x(t_1) = x_1$ – given

$$\text{s.t.} \quad \dot{x} = Ax + Bu$$
$$x(t_0) = x_0$$

Let $P(t)$ be the solution the DRE with $P(t_1) = 0$.

Compare $J(u) - x_0^T P(t_0) x_0$

$$\Rightarrow \quad J(u) - x_0^T P(t_0) x_0 = \int_{t_0}^{t_1} (u + R^{-1}B^T P x)^T R (u + R^{-1}B^T P x) \, dt$$

if with $u^* = -R^{-1}B^T P x(t)$, we have $x(t_1) = x_1$ for

$$\dot{x} = (A - BR^{-1}B^T P) x$$

then u^* would be the optimal control.

In general we let $u^* = -R^{-1}B^T P x + v(t)$

$$\Rightarrow \quad J(u) - x_0^T P(t_0) x_0 = \int_{t_0}^{t_1} v^T(t) R v(t) \, dt$$

$$\Rightarrow \quad \min \int_{t_0}^{t_1} v^T(t) R v(t) \, dt$$

$$\text{s.t.} \quad x(t_1) = x_1$$

$$\text{for} \quad \dot{x} = \underbrace{(A - BR^{-1}B^T P)}_{A_K(t)} x(t) + B v(t)$$
$$x(t_0) = x_0$$

Now assume $R = I \Rightarrow \min \int_{t_0}^{t_1} \|v(t)\|^2 \, dt$

and assume (A, B) is reachable

$$\Rightarrow \quad v^*(t) = B^T \Phi_K^T(t_1, t) W_K^{-1}(t_0, t_1)(x_1 - \Phi_K(t_1, t_0)x_0)$$

$$\Rightarrow \quad \underbrace{u^* = -R^{-1}B^T P x(t) + B^T \Phi_K^T(t_1, t) W_K^{-1}(t, t_1)(x_1 - \Phi_K(t_1, t)x(t))}$$

LQ control over infinite time interval

Consider

$$\dot{x} = Ax + Bu$$
$$x(0) = x_0$$

and $J(u) = \int_0^{t_1} (x^T Q x + u^T R u)\, dt$

We discuss when the optimal control exists and how to find it as $t_1 \to \infty$.

We know $P(t)$ for

$$\begin{cases} \dot{P} = -A^T P - PA + PBR^{-1}B^T P - Q \\ P(t_1) = 0 \end{cases}$$

Can be expressed as $P(t - t_1)$ $\hat{P}(t_1 - t)$

and $V(x_0, 0) = x_0^T P(0 - t_1) x_0$ is the optimal cost.

as $t_1 \to \infty$, $\Rightarrow$

$$J(u) := \int_0^\infty (x^T Q x + u^T R u)\, dt$$

If <u>(A, B) is reachable</u>, then we can let $u_T(t)$ $0 \le t \le T$ be the control that moves $x(t)$ from x_0 to 0 at $t = T$.

and then let $u(t) \equiv 0$. i.e.

$$\hat{u}(t) = \begin{cases} u_T(t) & 0 \le t \le T \\ 0 & t > T \end{cases}$$

$\Rightarrow$ $J(u) < \infty$.

1. we show $\lim_{t_1 \to \infty} P(0 - t_1) = P_\infty < \infty$

Note that $P(0-t_2) \ge P(0-t_1)$ $\forall\, t_2 \ge t_1$

i.e. $P(0-t_2) - P(0-t_1) \ge 0$.

$\to$ nondecreasing in t_1.

and $\forall\, t_1 \ge T$, $x_0^T P(0-t_1) x_0 \le \int_0^T (x^T Q x + \hat{u}^T(t) R \hat{u})\, dt$

$< \infty$

$\Rightarrow$ $P(0-t_1) < \infty$ $\forall\, t_1 > 0$.

$\lim_{t_1 \to \infty} x_0^T P(0-t_1) x_0$ exists $\forall\, x_0$

$\Rightarrow$ $\lim_{t_1 \to \infty} P(0-t_1) := P_\infty$ exists.

and $\quad P\rho \geq 0.$