# Observers

Consider

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$$
$$y = Cx \qquad y \in \mathbb{R}^p$$

If $x(t)$ is available, we can design $u = Kx$ to solve ex. pole placement.

However, if only $y(t) = Cx(t)$ is available, what can we do?

1. use $u = Ly$ output feedback
   $= LCx$

2. estimate first $x(t)$ based on $y(t)$ ( $\hat{x}(t)$ to denote the estimation ), then use $u = K\hat{x}$.

   a. how to estimate $x(t)$? Assume $(C, A)$ is observable.

   — we estimate $x(0)$. $\Rightarrow$ $\hat{x}(t)$ ✗

   — nonrobust

   b. We estimate $x(t)$ online $\Rightarrow$ the best we can have is $\hat{x}(t) \to x(t)$ as $t \to \infty$.

   $$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t))$$

   — observer

   we can just assume $\hat{x}(0) = 0$

our goal is to design $L_{n\times p}$ such that

$$\tilde{x}(t) = x(t) - \hat{x}(t) \to 0 \quad \text{as } t\to\infty.$$

$$\Rightarrow \quad \dot{\tilde{x}} = \dot{x} - \dot{\hat{x}}$$

$$= Ax + Bu - (A\hat{x} + Bu + L(Cx - C\hat{x}))$$

$$= \underbrace{(A - LC)}\tilde{x}$$

$\Rightarrow$ we need to find $L$ s.t. $A-LC$ is a stable matrix!

Since $(A-LC)^T = A^T - C^T L^T$ has the same eigenvalues of $A-LC$.

$\Rightarrow \exists L$ s.t. $A^T - C^T L^T$ is a stable matrix?

If we let $\bar{A} = A^T$, $\bar{B} = C^T$, $\bar{K} = -L^T \Rightarrow$

$\bar{A}^T - C^T L^T = \bar{A} + \bar{B}\bar{K}$ !

Solvable if $(A^T, C^T)$ is reachable, i.e.

$[C^T \ A^T C^T \ \cdots \ (A^T)^{n-1} C^T]$ has full rank

i.e. $\Omega = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ has full rank

$\Leftrightarrow (C, A)$ is observable!

then we can use $u = K\hat{x}$ as control.

$$\Rightarrow \begin{cases} \dot{x} = Ax + BK\hat{x} \\ \dot{\hat{x}} = A\hat{x} + BK\hat{x} + L(Cx - C\hat{x}) \end{cases} \qquad (\dot{x} = Ax + BKx)$$

we use $x$, $\tilde{x} = x - \hat{x}$ as the new coordinates

$$\dot{x} = Ax + BK(x - \tilde{x})$$
$$= (A+BK)x - BK\tilde{x}$$
$$\dot{\tilde{x}} = (A - LC)\tilde{x}$$

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} A+BK & -BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}$$

eigenvalues are eigenvalues of $A+BK$ and eigenvalues of $A-LC$

— Principle of separation: design state feedback and observer independent.

# Linear-Quadratic optimal control

Consider

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n$$
$$x(t_0) = x_0 \qquad u \in \mathbb{R}^m$$

Find the control on the time interval $[t_0, t_1]$ such that the following functional (Cost) is minimized

$$J(u) = \int_{t_0}^{t_1} [x(t)^T Q x(t) + u(t)^T R u(t)] \, dt + x(t_1)^T S x(t_1)$$

$Q \geq 0, \ R > 0, \ S \geq 0.$

Let $\quad V(x,t) = x^T P(t) x$

Ansatz and we conjecture $V$ is the optimal cost for $t_0 = t$, $t \in [t_0, t_1)$, and $x(t_0) = x$,

$$\frac{d V(x(t),t)}{dt} = \dot{x}(t)^T P(t) x + x^T(t) \dot{P} x(t) + x^T(t) P(t) \dot{x}(t)$$

$$= (Ax + Bu)^T P x + x^T \dot{P} x + x^T P (Ax + Bu)$$

$$= x^T (A^T P \qquad + PA + \dot{P}) x + u^T B^T P x + x^T P B u$$

$\Rightarrow$

$$V(x(t_1), t_1) - V(x_0, t_0)$$

$$= \int_{t_0}^{t_1} [x^T(t)(A^T P + PA + \dot{P})x(t) + u^T(t) B^T P x(t) + x^T(t) P B u(t)]\, dt$$

$$J(u) - V(x_0, t_0)$$

$$= \int_{t_0}^{t_1} [x^T(t)(A^T P + PA + Q + \dot{P})x + u^T R u + u^T B^T P x + x^T P B u]\, dt$$

$$x^T(t_1) S x(t_1) - x^T(t_1) P(t_1) x(t_1) \geq 0$$

$$P(t_1) = S$$

$$\left( u r u + u B P x + x P B u = r(u^2 + u r^{-1} B P x + x r^{-1} P B u) \right.$$

$$\left. = r(u + r^{-1} B P x)^2 - x P B r^{-1} B^T x \right)$$

$$= \int_{t_0}^{t_1} [x^T(t)(A^T P + PA + Q - PBR^{-1}B^T P + \dot{P})x(t) + (u + R^{-1}B^T P x)^T R (u + R^{-1} B^T P x)]\, dt$$

$$\Rightarrow \begin{cases} \dot{P} = -A^T P - PA + PBR^{-1}B^T P - Q \\ P(t_1) = S \end{cases} \quad \text{— Dynamical Riccati equation.}$$

$$\Rightarrow J(u) - V(x_0, t_0) = \int_{t_0}^{t_1} (u + R^{-1}B^T P x)^T R (u + R^{-1} B^T P x)\, dt$$

$$\geq 0$$

Furthermore $J(u) = V(x_0, t_0) = x_0^T P(t_0) x_0$

if $u = -R^{-1}B^T P X$

$\Rightarrow$ $u = -R^{-1} B^T P(t) X(t)$ is the optimal control

and $x_0^T P(t_0) x_0$ is the optimal cost!