Minimal realization

Consider $R(s)$ and its realization

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

(A, B, C) is minimal iff the system is both reachable and observable

$$\delta(R) = \text{rank } H_r = \text{rank} \begin{bmatrix} R_1 & \cdots & R_r \\ \vdots & & \\ R_r & \cdots & R_{2r-1} \end{bmatrix}$$

Example: $R(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{s+1} \\ \frac{-1}{s^2+3s+2} & \frac{1}{s+2} \end{bmatrix}$

$$\chi(s) = s^2+3s+2, \Rightarrow r=2$$

$$\Rightarrow \quad H_2 = \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & -1 & -2 \\ -1 & -2 & 1 & 2 \\ -1 & -2 & 3 & 4 \end{bmatrix}$$

$$\text{rank } H_2 = 3$$

Now suppose both

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n$$
$$y = Cx$$

and $\quad \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u \qquad \tilde{x} \in \mathbb{R}^n$

$y = \tilde{C}\tilde{x}$

are minimal realizations of $R(s)$.

question: how are these two systems are related?

Theorem: Such two realizations are linked by a linear transformation $\tilde{x} = Tx$.

Namely, $\tilde{A} = TAT^{-1}$, $\tilde{B} = TB$, $\tilde{C} = CT^{-1}$

Consequently, $\tilde{P} = [\tilde{B}\ \tilde{A}\tilde{B} \cdots \tilde{A}^{n-1}\tilde{B}]$

$$= [TB\ TAB \cdots TA^{n-1}B]$$

$$= TP$$

$$\tilde{\Omega} = \begin{bmatrix} \tilde{C} \\ \vdots \\ \tilde{C}\tilde{A}^{n-1} \end{bmatrix} = \Omega T^{-1}$$

$$\tilde{P} = TP, \quad \tilde{\Omega} = \Omega T^{-1}$$

The key to show the existence of T is to use the fact

$$\Omega P = \tilde{\Omega}\tilde{P} = H_n = \begin{bmatrix} R_1 & \cdots & R_n \\ \vdots & & \vdots \\ R_n & \cdots & R_{2n-1} \end{bmatrix}$$

(Since $R_i = CA^{i-1}B = \tilde{C}\tilde{A}^{i-1}\tilde{B}$)

and $\Omega A P = \tilde{\Omega}\tilde{A}\tilde{P}$

Characteristic polynomial of $R(s)$

Def: the characteristic polynomial $\rho(s)$ of $R(s)$ is the least common denominator of all minors of $R(s)$.

Example: $R_1(s) = \begin{bmatrix} \frac{a}{s+2} & \frac{1}{s+2} \\ \frac{1}{s+2} & \frac{1}{s+2} \end{bmatrix}$

minors: all elements in $R_1(s)$.

and $\frac{a-1}{(s+2)^2}$.

if $a \neq 1$, $\quad p(s) = (s+2)^2$

if $a = 1$, $\quad p(s) = s+2$

$$R_2(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{s+1} \\ \frac{-1}{s^2+3s+2} & \frac{1}{s+2} \end{bmatrix}$$ * *

minors: all elements

and $\frac{1}{(s+1)(s+2)} + \frac{2}{(s+1)^2(s+2)} = \frac{s+3}{(s+1)^2(s+2)}$

$\Rightarrow \quad p(s) = (s+1)^2(s+2)$

Def: The degree of $p(s)$ is called the degree of $R(s)$, - $\deg R(s)$.

Thm: $\delta(R) = \deg R$.

Given a scalar transfer function

$$r(s) = \frac{c_{p+1} s^p + \cdots + c_1}{s^n + a_n s^{n-1} + \cdots + a_1} \qquad p < n$$

$$= \frac{n(s)}{d(s)}$$

standard reachable realization:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ 0 & & & \ddots & 1 \\ -a_1 & \cdots & & & -a_n \end{bmatrix} x + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \qquad x \in \mathbb{R}^n$$

$$y = [c_1 \ \cdots \ c_{p+1} \ 0 \cdots 0]\, x$$

$$r(s) \frac{s+k}{s+k} = r(s) = \frac{\tilde{n}(s)}{\tilde{d}(s)} = \frac{n(s)(s+k)}{\underbrace{d(s)(s+k)}_{\tilde{\chi}(s)}}$$

$$\Rightarrow \quad \dot{\tilde{x}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ 0 & & & \ddots & 1 \\ -\tilde{a}_1 & \cdots & & & -\tilde{a}_{\tilde{n}} \end{bmatrix} \tilde{x} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [\tilde{c}_1 \ \cdots \ \tilde{c}_{\tilde{p}+1} \ 0 \cdots 0]\, \tilde{x}$$

$\Rightarrow$ Not observable!

State Feedback

Consider

$$\dot{x} = Ax + Bu \quad , \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m$$

we let $u = \underbrace{Kx}_{\text{feedback control}} + v(t)$ } feed forward control

Plug in u into the system

$$\dot{x} = (A+BK)x + Bv$$

$\dot{x} = (A+BK)x$ is called the closed-loop system.

Observation: reachable subspace $\mathrm{Im}\, P$ is invariant under state feedback. i.e.

$$\mathrm{Im}\,[B \ AB \cdots A^{n-1}B] = \mathrm{Im}\,[B \ (A+BK)B \cdots (A+BK)^{n-1}B] \quad \forall K \quad := \mathrm{Im}\,\hat{P}$$

Since $\mathrm{Im}\,(A+BK)B = \mathrm{Im}\,(AB + BKB)$

$$= \mathrm{Im}\,(AB) + \mathrm{Im}\,(BKB)$$

$$\subset \mathrm{Im}\,(AB) + \mathrm{Im}\,(B)$$

$$\subset \mathrm{Im}\,(B \ AB)$$

$$\Rightarrow \quad \mathrm{Im}\,(B\ (A+BK)B \ \cdots) \subset \mathrm{Im}\,(B\ AB \ \cdots)$$

$$\Rightarrow \quad \mathrm{Im}\,\hat{P} \subset \mathrm{Im}\,P$$

Let $A = (A+BK) - BK = \hat{A} - BK$

$$\Rightarrow \quad \mathrm{Im}\,P \subset \mathrm{Im}\,\hat{P}$$

$$\Rightarrow \quad \boxed{\mathrm{Im}\,P = \mathrm{Im}\,\hat{P}}$$

question: if we introduce $y = Cx$

and let $\hat{\Omega} = \begin{bmatrix} C \\ C(A+BK) \\ \vdots \\ C(A+BK)^{n-1} \end{bmatrix}$

$$\ker \Omega \overset{?}{=} \ker \hat{\Omega}$$

Now we discuss under what conditions:

$\exists$ $u = Kx$, s.t. $A+BK$ is asym. stable.

More precisely, we discuss under what conditions the eigenvalues of $(A+BK)$ can be assigned arbitrarily by K.

— pole placement problem:

Given $\varphi(s) = (s+s_1)\cdots(s+s_n)$

$$= s^n + \gamma_1 s^{n-1} + \cdots + \gamma_n$$

Find $u = Kx$, s.t.

$$\det(sI-(A+BK)) = s^n+\gamma_1 s^{n-1}+\cdots+\gamma_n$$

Thm: The pole placement problem is solvable iff (A,B) is reachable (Controllable), i.e. $\operatorname{Im} P = \mathbb{R}^n$.

Proof: we first consider the case $m=1$. and show sufficiency.

Assume first

$$A=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ 0 & & & & 1 \\ -a_n & & \cdots & & -a_1 \end{bmatrix}, \quad B=\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \qquad (**)$$

$$\Rightarrow \quad P=\begin{bmatrix} 0 & 0 & & 1 \\ \vdots & \vdots & \cdot^{\cdot^{\cdot}} & 0 \\ 0 & 1 & & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}_{n\times n} \Rightarrow \operatorname{Im} P = \mathbb{R}^n$$

$$\Rightarrow \quad \boxed{\det(sI-A) = s^n+a_1 s^{n-1}+\cdots+a_n}$$

let $u = Kx = k_1x_1+\cdots+k_nx_n = \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix}^T x$

$$\Rightarrow \quad A+BK = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & \cdots & & 1 \\ -a_n+k_1 & & \cdots & & -a_1+k_n \end{bmatrix}$$

$$\Rightarrow \det(sI-(A+BK)) = s^n+(a_1-k_n)s^{n-1}+\cdots+a_n-k_1$$

$$\det(sI-(A+BK)) = \varphi(s)$$

$$\Rightarrow \quad a_1-k_n=\gamma_1 \qquad \boxed{k_n = a_1-\gamma_1}$$

$$\vdots$$

$$a_n - k_1 = \gamma_n \qquad \boxed{k_1 = a_n - \gamma_n}$$

next, we just need to show any reachable (A, B) can be converted into $(**)$ by linear transformation.