Kalman decomposition and minimal realization

Let (A, B, C) $\begin{bmatrix} \dot{x} = Ax + Bu \\ y = Cx \end{bmatrix}$ be

a realization for $R(s)$.

Example: $R(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{s+2} \\ \frac{-1}{(s+1)(s+2)} & \frac{1}{s+2} \end{bmatrix}$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & -3 & 0 \\ 0 & -2 & 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 4 & 1 & 2 \\ -1 & 1 & 0 & 1 \end{bmatrix}.$$

reachable but not observable.

question: is the realization minimal?
any connection to reachability
and observability?

Recall realization $\Leftrightarrow$
$$C(sI - A)^{-1}B = R(s)$$

Given a realization (A, B, C)

we compute $\mathcal{R} = \operatorname{Im} P = \operatorname{Im} P(B \cdots A^{k-1}B)$

$$\ker \Omega = \ker \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

$\Rightarrow \quad \mathcal{R} \cap \ker \Omega$, $\Rightarrow \mathcal{R} = \mathcal{R} \cap \ker \Omega + V_{or}$

$\Rightarrow \ker \Omega = \mathcal{R} \cap \ker \Omega + V_{\bar{o}r}$

$$R^n = \underbrace{\mathcal{R} \cap \overbrace{\ker \Omega + V_{or}}^{}}_{\mathcal{R}} \overbrace{ + V_{\bar{o}\bar{r}}}^{\ker \Omega} + V_{o\bar{r}}$$

Then we can let $x = T\bar{x}$ $\quad \bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \\ \bar{x}_4 \end{bmatrix}$

Where $T = [\text{basis vectors of } \mathcal{R} \cap \ker \Omega \cdots \text{basis vectors of } V_{o\bar{r}}]$

$\Rightarrow \quad \bar{A} = T^{-1}AT, \ \bar{B} = T^{-1}B, \ \bar{C} = CT$

$$\begin{cases} \bar{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix} \\ \bar{C} = [0 \ \ C_2 \ \ 0 \ \ C_4] \end{cases}$$

In the new coordinates,

$$\mathcal{R} = \begin{bmatrix} * \\ * \\ 0 \\ 0 \end{bmatrix}, \quad \ker \Omega = \begin{bmatrix} * \\ 0 \\ * \\ 0 \end{bmatrix}$$

Recall that $\mathcal{R}$ and $\ker \Omega$ are A-invariant, i.e. $A\mathcal{R} \subset \mathcal{R}$, ...

$$\Rightarrow \quad \bar{A}\mathcal{R} \subset \mathcal{R} \quad \Rightarrow \quad \bar{A}\begin{bmatrix} * \\ * \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} + \\ + \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad B_3 = 0, \quad B_4 = 0.$

$\Rightarrow \quad A_{31} = 0 \quad A_{32} = 0, \quad A_{41} = 0, \quad A_{42} = 0$

$\ker \Omega = \begin{bmatrix} * \\ 0 \\ * \\ 0 \end{bmatrix} \Rightarrow \quad C_1 = 0, \quad C_3 = 0.$

Since $\bar{A} \begin{bmatrix} * \\ 0 \\ * \\ 0 \end{bmatrix} = \begin{bmatrix} + \\ 0 \\ + \\ 0 \end{bmatrix}$

$\Rightarrow \quad A_{21} = 0, \quad A_{43} = 0, \quad A_{23} = 0.$

$$\Rightarrow \quad R(s) = C(sI - A)^{-1} B$$
$$= C T T^{-1} (sI - A)^{-1} T T^{-1} B$$
$$= \bar{C} (sI - \bar{A})^{-1} \bar{B}$$

$$\Rightarrow \quad R_i = C A^{i-1} B = \bar{C} \bar{A}^{i-1} \bar{B}, \quad i = 1, \cdots, \infty$$

— Markov parameters.

$$R(s) = R_1 s^{-1} + R_2 s^{-2} + \cdots + \cdots$$

$$C(sI - A)^{-1} B = CBs^{-1} + CABs^{-2} + \cdots + CA^{i-1}Bs^{-i} + \cdots$$

$$\Rightarrow \quad \bar{C}\bar{B} = C_2 B_2, \quad \bar{C}\bar{A}\bar{B} = C_2 A_{22} B_2$$

$$\Rightarrow \quad \bar{C}\bar{A}^{i-1}\bar{B} = C_2 A_{22}^{i-1} B_2$$

$$\Rightarrow \quad R_i = C_2 A_{22}^{i-1} B_2, \quad i = 1, \cdots, \infty$$

$$\Rightarrow \quad R(s) = C_2 (sI - A_{22})^{-1} B_2 \quad !$$

$\Rightarrow$ (A_{22}, B_2, C_2) is also a realization of $R(s)$!

$$\begin{bmatrix} \dot{x} = A_{22}x + B_2 u \\ y = C_2 x \end{bmatrix}$$

and (A_{22}, B_2) is reachable, (C_2, A_{22}) is observable!

$\Rightarrow$ Being reachable and observable is a necessary condition for being a minimal realization.

We will show next this condition is also sufficient!

Definition: The McMillan degree $\delta(R)$ of $R(s)$ is the dimension of a minimal realization.

Suppose we have two realizations (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ of $R(s)$ $\Rightarrow$

$$\underline{R_i = CA^{i-1}B = \bar{C}\bar{A}^{i-1}\bar{B}}, \quad i=1, \cdots, \infty$$

We define the following Hankel matrix

$$H_i = \begin{bmatrix} R_1 & \cdots & R_i \\ \vdots & & \vdots \\ R_i & \cdots & R_{2i-1} \end{bmatrix}$$

$$= \begin{bmatrix} CB & \cdots & CA^{i-1}B \\ \vdots & & \\ CA^{i-1}B & & CA^{2i-2}B \end{bmatrix}$$

$$\Rightarrow \quad H_i = \begin{bmatrix} C \\ \vdots \\ CA^{i-1} \end{bmatrix} [B \cdots A^{i-1}B]$$

$$= \Omega_i P_i$$

$$\Rightarrow \quad \Omega = \Omega_n, \quad P = P_n$$

Now suppose (A, B, C) is both reachable and observable, dim $A = n$.

$\Rightarrow$ Ω_n and P_n have full rank n.

$\Rightarrow$ $(\Omega_n^T \Omega_n)_{n \times n}$ is nonsingular

$(P_n P_n^T)_{n \times n}$ is nonsingular

$\Rightarrow$ rank H_n = rank $\Omega_n P_n \leq$ rank $\Omega_n = n$

Meanwhile,

$$\Omega_n^T \Omega_n P_n P_n^T$$

$$= (\Omega_n^T \Omega_n)(P_n P_n^T) \quad \text{has rank } n$$

i.e. rank $\Omega_n^T H_n P_n^T = n$

$\Rightarrow$ rank $H_n \geq$ rank $\Omega_n^T H_n P_n^T = n$

$\Rightarrow$ $\boxed{\text{rank } H_n = n}$

Now suppose $(\tilde{A}, \tilde{B}, \tilde{C})$ is another realization of $R(s)$ of dimension $\tilde{n}$, $\Rightarrow$

$$\tilde{\Omega}_n \tilde{P}_n = H_n = \Omega_n P_n$$

$$\Rightarrow n = \text{rank } H_n = \text{rank } \tilde{\Omega}_n \tilde{P}_n$$

$$\leq \text{rank } \tilde{\Omega}_n \leq \tilde{n}$$

$$\left[\tilde{\Omega}_n = \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \vdots \\ \tilde{C}\tilde{A}^{n-1} \end{bmatrix} \right]$$

$\Rightarrow$ $\boxed{\tilde{n} \geq n}$

$\Rightarrow$ Sufficiency is proven!

The question then becomes: given $R(s)$, Can we determine $\delta(R)$ without obtaining any realization (A, B, C) first?

Recall H_i:

$$H_i = \begin{bmatrix} R_1 & \cdots & R_i \\ \vdots & & \vdots \\ R_i & \cdots & R_{2i-1} \end{bmatrix}$$

and $\boxed{R_{r+j} = -a_1 R_{r+j-1} - a_r R_j}$

$\Rightarrow$ $\boxed{\text{rank } H_i \leq \text{rank } H_r \quad \forall i \geq r}$

where $\chi(s) = s^r + a_1 s^{r-1} + \cdots + a_r$

— least common denominator of $R(s)$.

$$H_{r+1} = \begin{bmatrix} R_1 & \cdots & R_r & R_{r+1} \\ \vdots & & & \\ R_r & \cdots & R_{2r-1} & R_{2r} \\ R_r & & \cdots & R_{2r+1} \end{bmatrix}$$

i.e rank H_i reaches maximum at $i = r$!

Claim: $\boxed{\delta(R) = \text{rank } H_r}$

We only need to show $n \geq r$, where n is the dimension of a reachable and observable realisation, $\Rightarrow$

$$\delta(R) = n = \operatorname{rank} H_n$$

since $R(s) = C(sI-A)^{-1}B \Rightarrow$

$\det(sI-A)$ is a common denominator of $R(s)$,

and $\chi(s)$ is the least of all common denominators, thus

$$r = \deg \chi(s) \leq \deg \det(sI-A) = n$$

$\Rightarrow \quad n \geq r$,

$\Rightarrow \quad \operatorname{rank} H_n = \operatorname{rank} H_r = n = \delta(R)$