Stability (Cont'd)

Consider
$$\dot{x} = Ax \qquad x \in \mathbb{R}^n$$
$$x(0) = x_0$$

Assumptotically stable : $x(t) \to 0$ as $t \to \infty$ $\forall x_0 \in \mathbb{R}^n$

Stable : $\|x(t)\| < \infty$ $\forall t \geq 0$, $\forall x_0 \in \mathbb{R}^n$

Since $\quad x(t) = e^{At} x_0$.

Asym. stable $\Leftrightarrow$ $e^{At} \to 0$ as $t \to \infty$

Stable $\Leftrightarrow$ $\|e^{At}\| < K < \infty$.

Jordan form : $A = TJT^{-1}$, $e^{At} = Te^{Jt}T^{-1}$

$$J = \begin{bmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_r \end{bmatrix}, \qquad J_v = \begin{bmatrix} \lambda_v & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & \lambda_v \end{bmatrix}_{d_v \times d_v}$$

$$e^{J_v t} = e^{\sigma_v t}(\cos w_v t + j \sin w_v t)(I + tS + \cdots + \frac{t^{d_v - 1}}{(d_v - 1)!} S^{d_v - 1})$$

where, $\lambda_v = \sigma_v + j w_v$, $\quad S = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & 0 \end{bmatrix}_{d_v \times d_v}$

$\Rightarrow$ asym. stable $\Leftrightarrow$ $\sigma_v < 0$, $v = 1, \cdots, r$

unstable $\Leftarrow$ $\exists\, \sigma_{v^*} > 0$.

Stable ? $\Leftrightarrow$ no eigenvalue with positive real part, $\Rightarrow \sigma_v \leq 0$, $\forall v$.

Furthermore, for eigenvalues on

the imaginary axis, we must have

$d_v = 1$, i.e. $J_v = \{\lambda_v\}$, $\forall \lambda_v$ s.t.

$\sigma_v = 0$.

Example: $\dot{x} = Ax$

1. $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad \lambda_1 = \lambda_2 = 0$

2. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. $\quad \lambda_1 = \lambda_2 = 0$.

Study stability:

1. stable

2. unstable since $d = 2$.

Input-output stability

Consider

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

Let us first assume $\|u(t)\| < M \ \forall t \geq 0$.

we can assume $M = 1$.

When $\|y(t)\|$ is also bounded?

We assume $x(0) = 0$.

$$\Rightarrow \quad y(t) = \int_0^t C e^{A(t-s)} B u(s) ds$$

Since $\|y(t)\| = \|\int_0^t Ce^{A(t-s)}Bu(s)ds$

$$\leq \int_0^t \|Ce^{A(t-s)}Bu(s)\| ds$$

$$\leq \|C\| \int_0^t \|e^{A(t-s)}\| ds \, \|B\| M$$

$(M=1)$

$\Rightarrow$ a sufficient condition for input-output stability is that A is a stable matrix, i.e. A has all eigenvalues with negative real parts.

If (A, B) is reachable and (C, A) is observable, then

input-output stability $\Leftrightarrow$ A is a stable matrix

Now we assume in general

$u(t) \in L_p(0,\infty)$. i.e.

$$\int_0^\infty \|u(s)\|^p ds < \infty$$

When $p=2$, $\int_0^\infty u^T(s)u(s)\,ds < \infty$

$\Rightarrow$ $y(t) \in L_p[0,\infty)$ $\forall$ $u(t) \in L_p[0,\infty)$ if

A is a stable matrix

The Lyapunov equation and Stability

Consider a scalar function

$$V(x) = x^T P x \quad - \text{ energy function}$$

where P is positive definite.

Recall $P_{n\times n}$ is said to be positive definite if $P^T = P$ and all eigenvalues of P are positive.

$$\Rightarrow \quad P = L J L^T, \quad L L^T = I$$

where $J = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$, $\lambda_i > 0$, $i = 1, \cdots, n$

$$\Rightarrow \quad V(x) = x^T L J L^T x$$

$$= z^T J z \quad \text{where } z = L^T x$$

$$= z^T \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} z, \quad \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$$

$$\underbrace{\lambda_1 z^T z}_{= \lambda_1 \|x\|^2} \le V(x) \le \underbrace{\lambda_n z^T z}_{= \lambda_n (x^T L L^T x) = \lambda_n \|x\|^2}$$

$$\Rightarrow \boxed{\lambda_1 \|x\|^2 \leq x^T P x \leq \lambda_n \|x\|^2}$$

$\lambda_1 = \min \text{eig}(P)$, $\lambda_n = \max \text{eig}(P)$.

For a $P > 0$, we have

$$\boxed{\begin{aligned} P &= L J L^T = L J^{1/2} J^{1/2} L^T = R R^T \\ &= L J^{1/2} \underbrace{L^T L}_{I} J^{1/2} L^T = Q^{1/2} Q^{1/2} \end{aligned}}$$

If $V(x) = x^T P x$ $\quad P > 0$

$\Rightarrow V(x) > 0$ if $\|x\| \neq 0$.

and $\frac{dV}{dt} \leq 0$ "losing energy as t goes"

then the system may go back to the origin.

Since
$$\begin{aligned} \dot{V} &= \dot{x}^T P x + x^T P \dot{x} \\ &= (Ax)^T P x + x^T P A x \\ &= x^T (A^T P + P A) x \leq 0 \\ &= x^T (-Q) x = -x^T Q x \qquad Q > 0 \end{aligned}$$

$\Rightarrow \boxed{A^T P + P A = -Q}$ — Lyapunov equation

For any $Q \in \mathbb{R}^{n\times n}$, if A is a stable matrix,

$$P = \int_0^{\infty} e^{A^T t} Q e^{At} dt$$

is a solution to the Lyapunov equation.

Proof: let $L(t) = e^{A^T t} Q e^{At} \Rightarrow$

$$\dot{L} = A^T e^{A^T t} Q e^{At} + e^{A^T t} Q A e^{At}$$

$$= A^T L(t) + L(t) A$$

$$\Rightarrow \int_0^{\infty} \dot{L}(t) dt = A^T P + P A$$

$$A^T P + PA = L(\infty) - L(0) = -Q \quad !$$

In fact $\int_0^{\infty} e^{A^T t} Q e^{At} dt$ is the unique solution.

Thm: the following statements are equivalent:

1. $A^T P + PA = -C^T C$

   has solution $P > 0$. where (C, A) is observable.

2. A is a stable matrix.

Proof: "if". When A is a stable matrix.

$$P = \int_0^\infty e^{A^T t} C^T C e^{At} dt \overset{?}{>} 0$$

- observability Gramian $M(0,\infty)$

$\Rightarrow \quad P = M(0,\infty) > 0$.

"only if": assume

$A^T P + PA = -C^T C$ has $P > 0$.

then we let $V(x) = x^T(t) P x(t)$.

$\Rightarrow \quad \dot{V} = -x^T C^T C x = -\|Cx(t)\|^2 \leq 0, \ \forall x$

$\Rightarrow \quad V(x(t)) - V(x(0)) \leq 0$.

$\Rightarrow \quad x^T(t) P x(t) \leq x_0^T P x_0$

$\lambda_{min} \|x(t)\|^2 \leq x^T(t) P x(t) \leq x_0^T P x_0 \leq \lambda_{max} \|x_0\|^2$

$\Rightarrow \quad \|x(t)\|^2 \leq \frac{\lambda_{max}}{\lambda_{min}} \|x_0\|^2 < \infty, \ \forall t \geq 0$.

$\Rightarrow \quad \dot{x} = Ax$ is stable.

$\Rightarrow \quad \sigma_v \leq 0$.

If the system is not asym. stable,

$\Rightarrow$ $\exists$ eigenvalues on the imaginary axis.

$\Rightarrow$ $\exists$ periodic solution $x(t)$: i.e.

$x(t+T) = x(t)$ $\forall$ $t \geq 0$, $T > 0$.

$$\Rightarrow \quad V(x(t+T)) = V(x(t))$$

however since $\dot{V} = -\|Cx(t)\|^2$

$$\Rightarrow \quad Cx(t) \equiv 0 \quad \text{on } [0, T]$$

$$\Rightarrow \quad Cx^{(k)}(t) \equiv 0 \quad \text{on } [0, T], \; k = 0, 1, \cdots$$

Since $x^{(k)}(t) = A^k x(t)$

$$\Rightarrow \quad \begin{bmatrix} C \\ CA \\ \vdots \\ CA^k \\ \vdots \end{bmatrix} x(t) \equiv 0$$

$$\Rightarrow \quad \begin{bmatrix} C \\ \vdots \\ CA^{n-1} \end{bmatrix} x(t) \equiv 0$$

$$\Rightarrow \quad \Omega x(t) \equiv 0$$

$\Rightarrow$ $x(t) \equiv 0$ since (C, A) is observable.

$\Rightarrow$ the only periodic solution is $x(t) \equiv 0$.

$\Rightarrow$ A is a stable matrix!