Observability

Consider

$$\dot{x} = A(t)x + B(t)u(t) \qquad \text{W.L.O.G, we set}$$
$$y = C(t)x + D(t)u(t) \qquad u(t) = 0$$
$$x(t_0) = x_0$$

Observability Gramian:

$$M(t_0, t_1) = \int_{t_0}^{t_1} \Phi^T(s, t_0) C^T(s) C(s) \Phi(s, t_0)\, ds$$

Observable: iff M is nonsingular

i.e. $\ker M = 0$

In general, two initial states a, b generate the same output $y(t)$ iff $a - b \in \ker M$

Observability for time-invariant systems

$$(*) \quad \begin{aligned} \dot{x} &= Ax \qquad x \in \mathbb{R}^n \\ y &= Cx \qquad y \in \mathbb{R}^p \end{aligned}$$

where A, C are constant

Define

$$\Omega = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Thm: $(*)$ is observable iff

Ω has full column rank, or

$\ker \Omega = 0$

$$\Rightarrow \quad \boxed{\ker \Omega = \ker M}$$

where: $$M = \int_{t_0}^{t_1} e^{A^T(s-t_0)} C^T C e^{A(s-t_0)} ds$$

Recall reachability gramian

$$W(t_0,t_1) = \int_{t_0}^{t_1} e^{A(t_1-s)} B B^T e^{A^T(t_1-s)} ds$$

$$\overset{r=t_1-s}{=} \int_{t_1-t_0}^{0} e^{Ar} B B^T e^{A^T r}(-dr)$$

$$\boxed{= \int_0^{t_1-t_0} e^{Ar} B B^T e^{A^T r} dr}$$

$$M \overset{r=s-t_0}{=\!=} \boxed{\int_0^{t_1-t_0} e^{A^T r} C^T C e^{Ar} dr}$$

$A - A^T \qquad B - C^T$

Let $\bar{A} = A^T \quad \bar{B} = C^T$

$$\Rightarrow \quad M = \bar{W} = \int_0^{t_1 - t} e^{\bar{A}r} \bar{B}\bar{B}^T e^{\bar{A}^T r} dr$$

Recall $\quad \mathbb{R}^n = \operatorname{Im} W + \ker W \qquad P = [B \ AB \cdots A^{n-1}B]$

$$= \operatorname{Im} P + \ker P^T$$

$$= \operatorname{Im} \bar{W} + \ker \bar{W}$$

$$= \operatorname{Im} M + \ker M$$

$$= \operatorname{Im} \bar{P} + \ker \bar{P}^T$$

$$\Rightarrow \quad \ker M = \ker \bar{P}^T = \ker [\bar{B} \ \bar{A}\bar{B} \cdots \bar{A}^{n-1}\bar{B}]^T$$

$$= \ker \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \ker \Omega \quad !$$

We call $\ker \Omega$ the unobservable subspace ($\operatorname{Im} P$ is called the reachable subspace).

$y(t) = 0 \ \forall t \geq 0$ iff $x_0 \in \ker \Omega$

We can show $\ker \Omega$ is A-invariant

i.e. $\forall x_0 \in \ker \Omega, \Rightarrow Ax_0 \in \ker \Omega$

$$\Rightarrow \quad e^{At}x_0 \in \ker \Omega \ , \quad \text{if } x_0 \in \ker \Omega$$

$$\Rightarrow \quad y = Ce^{At}x_0 = 0$$

Duality between reachability and observability

Consider

$$\dot{x} = A(t)x + B(t)u$$

$$x(t_0) = a$$

$$\Rightarrow \quad W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_1, s)B(s)B^T(s)\Phi^T(t_1, s)\,ds$$

We want to define a system

$$\dot{z} = \bar{A}(t)z$$

$$y(t) = \bar{C}(t)z$$

$$z(\bar{t}_0) = b$$

$$\Rightarrow \quad \bar{M}(t_0, t_1) = \int_{t_0}^{t_1} \bar{\Phi}^T(s, \bar{t}_0)\bar{C}^T(s)\bar{C}(s)\bar{\Phi}(s, \bar{t}_0)\,ds$$

s.t. $\bar{M}(t_0, t_1) = W(t_0, t_1)$

$$\Rightarrow \quad \bar{t}_0 = t_1 \ , \quad \bar{C}(t) = B^T(t)$$

$$\Rightarrow \quad \bar{A}(t) = -A^T(t) \quad !$$

Please show $\bar{\Phi}^T(s, t_1) = \Phi(t_1, s)$

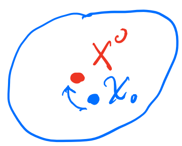
By choosing $\bar{t}_0 = t_1$, we have a system that moves backwards in time.

Stability

For a general dynamical system

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n, \quad \text{time-invariant}$$

We say x^0 is an equilibrium if $f(x^0) = 0$



Stability

Consider

$$(**) \quad \dot{x} = Ax \quad , \quad x \in \mathbb{R}^n$$

and take $x^0 = 0$ as the equilibrium.

Def: We say (**) is asymptotically stable if $x(t) \to 0$ as $t \to \infty$ $\forall\, x(t_0) \in \mathbb{R}^n$,

We say (**) is stable if $\|x(t)\| < \infty$ $\forall\, x(t_0) \in \mathbb{R}^n$.

We say (**) is unstable if it is not stable.

Since $x(t) = e^{A(t-t_0)} x(t_0)$

$\Rightarrow$ 1) we can assume $t_0 = 0$

2) $x(t) \to 0 \Rightarrow e^{At} \to 0$ as $t \to \infty$

3) $\|x(t)\| \leq \|e^{At}\| \, \|x(0)\| \Rightarrow$

$\|x(t)\| < \infty \iff \|e^{At}\| < \infty \quad \forall t \geq 0.$

Remark: for $\dot{x} = Ax + Bu$, if $\dot{x} = Ax$ is not asym. stable, we can let $u = Kx(t)$, $\Rightarrow \dot{x} = (A+BK)x$ s.t. $\dot{x} = (A+BK)x$ is asym. stable

— feedback stabilization

Thm: (**) is asym. stable iff all eigenvalues of A have negative real parts.

Example: $A = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$$\Rightarrow e^{At} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}$$

Proof: By Jordan decomposition:

$$A = TJT^{-1}$$

whee $$J = \begin{bmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_r \end{bmatrix}$$

$$J_\nu = \begin{bmatrix} \lambda_\nu & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_\nu \end{bmatrix}$$ $J_\nu = [\lambda_\nu]$ one dimension

$$\Rightarrow \quad e^{At} = T e^{Jt} T^{-1}$$

and $$e^{Jt} = \begin{bmatrix} e^{J_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{J_r t} \end{bmatrix}$$

$$\Rightarrow \quad e^{At} \Leftrightarrow e^{J_\nu t} \to 0 \quad \nu = 1, \cdots, r$$

Since $$J_\nu = \underbrace{\begin{bmatrix} \lambda_\nu & & 0 \\ & \ddots & \\ 0 & & \lambda_\nu \end{bmatrix}}_{\lambda_\nu I} + \underbrace{\begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & & 0 \end{bmatrix}}_{S_\nu}$$

and

$$\lambda_\nu I \cdot S_\nu = S_\nu \cdot \lambda_\nu I$$

$$\Rightarrow \quad e^{J_\nu t} = e^{(\lambda_\nu I t + S_\nu t)} = e^{\lambda_\nu I t} e^{S_\nu t}$$

$$= e^{\lambda_\nu t} e^{S_\nu t}$$

Since $S_\nu^{\nu} = 0$

$$\Rightarrow \quad e^{S_\nu t} = \left(I + S_\nu t + \cdots + S_\nu^{\nu-1} \frac{t^{\nu-1}}{(\nu-1)!} \right)$$

Let $\lambda_v = \sigma_v + j\omega_v$

$\Rightarrow \quad e^{\lambda_v t} = e^{\sigma_v t} e^{j\omega_v t} = e^{\sigma_v t}(\cos\omega_v t + j\sin\omega_v t)$

$\Rightarrow \quad e^{J_v t} = \underline{\underline{e^{\sigma_v t}}}\underbrace{(\cos\omega_v t + j\sin\omega_v t)\, e^{S_v t}}$

$\Rightarrow \quad e^{J_v t} \to 0 \quad \text{if} \quad \sigma_v < 0$

$e^{J_v t} \to \infty \quad \text{if} \quad \sigma_v > 0$

$\boxed{e^{J_v t} \to\ ? \quad \text{if} \quad \sigma_v = 0}$

$\Rightarrow$ $e^{At} \to 0$ as $t \to \infty$ iff all eigenvalues have negative real parts.

e^{At} diverges if at least one eigenvalue has positive real part $\Rightarrow$ unstable.