Reachability and observability

For a time invariant system

$$\dot{x} = Ax + Bu \quad , \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m$$

$$W(t_0, t_1) = \int_{t_0}^{t_1} e^{A(t_1 - s)} B B^T e^{A^T(t_1 - s)} \, ds$$

Let $P = [B \ AB \cdots A^{n-1}B]$

We have shown $\operatorname{Im} P = \operatorname{Im} W$

$\Rightarrow$ reachable iff $\operatorname{Im} P = \mathbb{R}^n$

denote $\mathcal{R} = \operatorname{Im} P$ — reachable subspace.

Recall that x_1 at t_1 can be reached from x_0 at t_0 iff $d = x_1 - e^{A(t_1 - t_0)} x_0 \in \operatorname{Im} W = \operatorname{Im} P := \mathcal{R}$

Lemma: $\mathcal{R}$ is A-invariant, i.e. $\forall x_0 \in \mathcal{R}$ $Ax_0 \in \mathcal{R}$.

Proof: $x_0 \in \mathcal{R} = \operatorname{Im} P \Rightarrow$

$$x_0 = [B \ AB \cdots A^{n-1}B] a$$

$$\Rightarrow \quad Ax_0 = [AB \ A^2B \cdots A^nB] a \in \operatorname{Im} P$$

By Cayley-Hamilton, $A^nB = \sum_{i=0}^{n-1} c_i A^i B$

We use $A\mathcal{R} \subset \mathcal{R}$ to denote this.

$$\Rightarrow \quad A^2 \mathcal{R} \subset \mathcal{R}$$

$$\Rightarrow \quad A^k \mathcal{R} \subset \mathcal{R}, \quad k = 0, 1, \cdots, \infty$$

$$\Rightarrow \quad e^{At}x_0 \in \mathcal{R} \qquad \forall x_0 \in \mathcal{R}, \ \forall t \in R$$

$$\Rightarrow \quad \underline{e^{At}\mathcal{R} \subset \mathcal{R}}$$

$$\Rightarrow \quad d \overset{?}{=} x_1 - e^{A\varepsilon}x_0 \qquad \underline{\varepsilon = t_1 - t_0}$$

$$\in \mathcal{R}$$

Thm: For any $x_0 \in \mathcal{R}$ and any $x_1 \in \mathcal{R}$

$\exists\, u(t)$, s.t. $x(\varepsilon) = x_1$, while $x(0) = x_0$

$\forall \varepsilon > 0$.

Proof: Since $e^{A\varepsilon}x_0 \in \mathcal{R}$ $\forall x_0 \in \mathcal{R}$.

and $x_1 \in \mathcal{R}$,

$$\Rightarrow \quad d = \underset{\in \mathcal{R}}{x_1} - \underset{\in \mathcal{R}}{e^{A\varepsilon}x_0} \in \mathcal{R} \quad (\mathrm{Im}\, P = \mathrm{Im}\, W)$$

$$(\quad d = (x_1^{R} + x_1^{\bar{R}}) - e^{A\varepsilon}(x_0^{R} + x_0^{\bar{R}})$$

$$= \underline{x_1^{R} - e^{A\varepsilon}x_0^{R}} + \underbrace{x_1^{\bar{R}} - e^{A\varepsilon}x_0^{\bar{R}}}_{0} \quad \in \mathcal{R}$$

State decomposition based on $\mathcal{R}$:

Consider $\dot{x} = Ax + Bu$

$$\mathcal{R} = \mathrm{Im}\,[B \cdots A^{n-1}B]$$

$$\Rightarrow \quad \mathbb{R}^n = \mathcal{R} + \mathcal{V}$$

$$x = T\tilde{x} = T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

$$\Rightarrow \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u$$

$(\dot{x}_2 = A_{22} x_2)$

$x_1 \sim \mathcal{R}$, and $[A_{11}, B_1]$ is reachable

i.e. $[B_1, A_{11}B_1, \cdots, A_{11}^{\bar{n}}B_1]$ has full rank

$\bar{n} = \dim \mathcal{R} - 1$.

The reason $A_{21} = 0$ is that $\mathcal{R}$ is A-invariant.

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \in \mathcal{R} \qquad \begin{bmatrix} \dot{x}_1 \\ 0 \end{bmatrix}$$

$A_{21} x_1$

$\Rightarrow A_{21} = 0$.

We can form T as:

$$T = [\overbrace{v_1 \cdots v_k}^{x_1} \ \overbrace{Q}^{x_2}]$$

where $v_1, \cdots, v_k$ are basis vectors for $\mathcal{R}$.

$$\Rightarrow \quad \tilde{A} = T^{-1} A T, \quad \tilde{B} = T^{-1} B$$

Observability

Consider

$$\dot{x} = A(t) x + B(t) u(t) \qquad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m$$

$$y = C(t) x + D(t) u(t) \qquad y \in \mathbb{R}^p$$

$p < n$

We assume $u(t)$ is known

Recall $x(t) = \Phi(t, t_0) x_0 + \int_{t_0}^{t} \Phi(t,s) B(s) u(s) ds$

$\Rightarrow$ $y(t) = C(t) x(t)$

$$= C(t)\Phi(t,t_0)x_0 + \int_{t_0}^{t} C(t)\Phi(t,s)B(s)u(s)ds$$

$$+ D(t)u(t)$$

$\Rightarrow$ (*) $\underbrace{C(t)\Phi(t,t_0)}_{p \times n} x_0 = v(t)$ — known

$t \in [t_0, t_1]$

Under what conditions can we determine x_0 uniquely ?

We reexpress (*) in more general form:

$\omega(t) x_0 = v(t)$ $\quad t \in [t_0, t_1]$

Recall for $W x_0 = v_p$ $\quad W_{p \times n}$

$\exists$ a unique solution (if solution exists)

1-1, iff W has full column rank !

Skinny matrix

Proposition: $\omega(t) x_0 = v(t)$, where

$\omega = [\omega_1(t), \cdots, \omega_n(t)]$

has unique solution iff $\omega_1(t), \cdots, \omega_n(t)$

are linearly independent on $[t_0, t_1]$, i.e.

$$\sum_{i=1}^{n} c_i w_i(t) = 0 \quad \forall\, t \in [t_0, t_1] \iff c_i = 0,\ i=1, \cdots, n.$$

Proof: "if": assume x_1, x_2 are solutions

i.e. $\quad w(t) x_1 = v(t) = w(t) x_2 \quad , t \in [t_0, t_1]$

$$\Rightarrow \quad w(t)(x_1 - x_2) = 0$$

$$\Rightarrow \quad x_1 - x_2 = 0 \quad \Rightarrow x_1 = x_2$$

"only if": if the solution is unique,

then $\quad w(t) a = 0 \quad \forall\, t \in [t_0, t_1]$

$$\iff \quad a = 0$$

otherwise $\quad w(t)(x_0 + a) = w(t) x_0 = v(t)$

$\Rightarrow$ $x_0 + a$ is also a solution.

Recall that $f_1(t), \cdots, f_n(t)$ are linearly independent

iff $\displaystyle\int_{t_0}^{t_1} \begin{bmatrix} f_1(s) \\ \vdots \\ f_n(s) \end{bmatrix} [f_1^T \cdots f_n^T]\, ds$ is nonsingular

Now we go back to

$$C(t) \Phi(t, t_0) x_0 = v(t)$$

$\Rightarrow$ the solution is unique iff columns of $C(t) \Phi(t, t_0)$ are linearly independent

i.e.

$$M(t_0,t_1) := \int_{t_0}^{t_1} \Phi^T(s,t_0) C^T(s) C(s) \Phi(s,t_0)\, ds$$

is nonsingular (positive definite)

— Observability Gramian

Thm: the system is observable iff $M(t_0,t_1)$ is nonsingular.

Proof: "if":

$$\Phi^T(t,t_0) C^T(t) C(t) \Phi(t,t_0) x_0 = \Phi^T(t,t_0) C^T v(t)$$

$$\Rightarrow \quad \underbrace{\int_{t_0}^{t_1} \Phi^T C^T C \Phi\, ds}_{M}\, x_0 = \int_{t_0}^{t_1} \Phi^T C^T v\, ds$$

$$\Rightarrow \quad \underline{x_0 = M(t_0,t_1)^{-1} \int_{t_0}^{t_1} \Phi(s,t_0) C^T(s) v(s)\, ds}$$

Thm: If $M(t_0,t_1)$ is singular, for any given $u(t)$, the two initial states x_0^1, x_0^2 will produce the same output $y(t)$ iff

$$x_0^1 - x_0^2 \in \ker M(t_0,t_1)$$

i.e. $M(t_0,t_1)(x_0^1 - x_0^2) = 0$.

or $M(t_0,t_1) x_0^1 = M(t_0,t_1) x_0^2$