Reachability (Continued)

Consider

$$\dot{x} = A(t)x + B(t)u$$

For $x(t_0) = x_0$, $x(t_1) = x_1$,

Let $\quad d = x_1 - \Phi(t_1, t_0)x_0$

Reachable (Controllable) $\Leftrightarrow$ $\forall d \in \mathbb{R}^n$, $\exists u(t)$, s.t

$$\int_{t_0}^{t_1} \Phi(t_1, s)B(s)u(s)ds = d$$

$\Leftrightarrow$ rows of $\Phi(t_1, t)B(t)$ are linearily independent over $[t_0, t_1]$

$\Leftrightarrow$ Reachability Gramian

$$W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_1, s)B(s)B^T(s)\Phi^T(t_1, s)ds$$

is nonsingular (positive definite)

$\Rightarrow \quad u^*(t) = B^T(t)\Phi^T(t_1, t)W^{-1}(t_0, t_1)d$

In general, $\exists u$, s.t. $x(t_1) = x_1$ is reachable from $x(t_0) = x_0$ iff

$$d = x_1 - \Phi(t_1, t_0)x_0 \in \operatorname{Im} W(t_0, t_1)$$

i.e. $\exists a$ s.t. $\quad d = W(t_0, t_1)a \quad a \in \mathbb{R}^n$

$$\Rightarrow \quad u^*(t) = B^T(t)\,\Phi^T(t_1,t)\,a$$

Energy of $u(t)$: $\quad \left(\int_{t_0}^{t_1} u^T(s)u(s)\,ds\right)^{1/2} = \left(\int_{t_0}^{t_1} \|u(s)\|^2\,ds\right)$

We show now $u^*(t)$ has the minimal energy among all $u(t)$ that brings x_0 to x_1 at t_1.

$$\int_{t_0}^{t_1} \Phi(t_1,s)B(s)u^*(s) = d = \int_{t_0}^{t_1} \Phi(t_1,s)B(s)u(s)\,ds$$

$$\Rightarrow \quad a^T\int_{t_0}^{t_1} \Phi(t_1,s)B(s)(u^*(s)-u(s))\,ds = 0$$

$$\Rightarrow \quad \int_{t_0}^{t_1} u^{*T}(s)(u^*(s)-u(s))\,ds = 0.$$

$$\Rightarrow \quad \int_{t_0}^{t_1} u^{*T}u^*\,ds = \int_{t_0}^{t_1} u^{*T}u\,ds$$

$$\leq \left(\int_{t_0}^{t_1} u^{*T}u^*\,ds\right)^{1/2}\left(\int_{t_0}^{t_1} u^Tu\,ds\right)^{1/2}$$

$$\Rightarrow \quad \left(\int_{t_0}^{t_1} u^{*T}u^*\,ds\right)^{1/2} \leq \left(\int_{t_0}^{t_1} u^Tu\,ds\right)^{1/2}$$

Example: Consider $\dot{x} = Ax + Bu$

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Study if the system is reachable.

$$e^{At} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

$$\Rightarrow \quad \Phi(t_1, t) B(t) = \Phi(t_1 - t) B$$

$$= \begin{bmatrix} e^{\lambda_1(t_1-t)} & 0 \\ 0 & e^{\lambda_2(t_1-t)} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{\lambda_1(t_1-t)} \\ e^{\lambda_2(t_1-t)} \end{bmatrix}$$

Either we show $e^{\lambda_1(t_1-t)}$, $e^{\lambda_2(t_1-t)}$ are linearly independent by definition on $[t_0, t_1]$

or we check if

$$\int_{t_0}^{t_1} \begin{bmatrix} e^{\lambda_1(t_1-s)} \\ e^{\lambda_2(t_1-s)} \end{bmatrix} \begin{bmatrix} e^{\lambda_1(t_1-s)} & e^{\lambda_2(t_1-s)} \end{bmatrix} ds$$

is nonsingular

linear independence $\Rightarrow \quad c_1 e^{\lambda_1 \Delta t} + c_2 e^{\lambda_2 \Delta t} = 0$.

$$\Rightarrow \quad c_1 = -c_2 e^{(\lambda_2 - \lambda_1)\Delta t}$$

1. if $\lambda_2 - \lambda_1 \neq 0$, $\Rightarrow c_2 = 0$, $\Rightarrow c_1 = 0$

   $e^{\lambda_1 \Delta t}$, $e^{\lambda_2 \Delta t}$ are linearly independent!

   $\Rightarrow$ reachable

2. if $\lambda_1 = \lambda_2$, $\Rightarrow \quad c_1 = -c_2$

$\Rightarrow e^{\lambda_1 \Delta t}, e^{\lambda_2 \Delta t}$ are linearly dependent.

$\Rightarrow$ not reachable

We will study time invariant systems after break:

$$(*) \quad \dot{x} = \underset{n\times n}{A} x + \underset{n\times m}{B} u \qquad x \in \mathbb{R}^n, \; u \in \mathbb{R}^m$$

Define $\Gamma := [\underset{n\times m}{B} \; AB \cdots A^{n-1}B]$

Thm: $(*)$ is reachable iff

$$\operatorname{Im} \Gamma = \mathbb{R}^n$$

$\big($ $\underline{d \in \operatorname{Im}\Gamma \text{ if } \exists\, a \in \mathbb{R}^{nm}, \text{ s.t. } d = \Gamma a}$ $\big)$

For time-invariant system, we

$$W(t_0, t_1) = \int_{t_0}^{t_1} e^{A(t_1 - s)} B B^T (e^{A(t_1 - s)})^T ds$$

and we know $(*)$ is reachable iff $W(t_0, t_1)$ is nonsingular!

Proof: We only need to show

$$\operatorname{Im}\Gamma = \operatorname{Im} W(t_0, t_1) \quad \forall\, t_1 > t_0$$

$\operatorname{Im}\Gamma$ is called reachable subspace

We need to show $\operatorname{Im} P \subseteq \operatorname{Im} W(t_0,t_1)$

and $\operatorname{Im} W(t_0,t_1) \subseteq \operatorname{Im} P$

1. $\operatorname{Im} P \subseteq \operatorname{Im} W(t_0,t_1)$

Recall for a matrix $P_{n\times k}$, we have

$$\mathbb{R}^n = \operatorname{Im} P \oplus \ker P^T$$

or $n = \dim \operatorname{Im} P + \dim \ker P^T$

($d \in \ker P^T$, if $P^T d = 0$)

$\Rightarrow$ $n = \dim \operatorname{Im} P + \dim \ker P^T$

$n = \dim \operatorname{Im} W + \dim \ker W^T$

$\Rightarrow$ we show equivalently $\ker W^T \subseteq \ker P^T$

i.e. $\forall\ a \in \ker W^T$ ($W^T a = 0$) we

$a \in \ker P^T$ ($P^T a = 0$)

Since $W^T = W$, $\Rightarrow$ $\ker W \subseteq \ker P^T$

Now let $a \in \ker W$, i.e.

$$a^T \int_{t_0}^{t_1} e^{A(t_1-s)} B B^T e^{A^T(t_1-s)} ds\, a = 0 .$$

$$\Rightarrow \int_{t_0}^{t_1} \| B^T e^{A^T(t_1-s)} a \|^2 ds = 0$$

$$\Rightarrow \quad B^T e^{A^T(t_1-t)} a = 0 \quad \forall\, t \in [t_0, t_1]$$

$$\Rightarrow \quad B^T\left(I + (t_1-t)A^T + \cdots + \frac{(t_1-t)^k}{k!}(A^T)^k + \cdots + \ldots\right)a = 0$$

$$B^T a \cdot 1 + B^T A^T a \underbrace{(t_1 - t)}_{\tilde{t}} + \cdots + B^T (A^T)^k a \frac{(t_1-t)^k}{k!} + \cdots = 0 .$$

Since $1, \tilde{t}, \tilde{t}^2, \cdots$ are linearly independent over $[0, \tilde{t}_1]$ $\tilde{t}_1 > 0$.

$$\Rightarrow \quad B^T a = 0,\ B^T A^T a = 0, \ldots,\ B^T (A^T)^k a = 0, \cdots B^T (A^T)^{n-1} a = 0,$$

due to Cayley-Hamilton theorem

i.e. for an $A_{n\times n}$, we have

$p_A(A) = 0$. where $p_A(s) = \det(sI - A)$

is the so-called characteristic polynomial for A. $p_A(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$

$$p_A(A) = 0 \Rightarrow A^n = -a_1 A^{n-1} - \cdots - a_n I$$

$$\Rightarrow \quad \begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T (A^T)^{n-1} \end{bmatrix} a = 0 \quad \text{i.e. } P^T a = 0 .$$

i.e. $a \in \ker P^T$

$$\Rightarrow \quad \ker W \subseteq \ker P^T$$

or $\quad \mathrm{Im}\, P \subseteq \mathrm{Im}\, W$

2. We need to show $\mathrm{Im}\, W \subseteq \mathrm{Im}\, P$

$$d \in \operatorname{Im} W \Rightarrow \exists\, a \quad s.t. \quad d = Wa$$

$$d = \int_{t_0}^{t_1} e^{A(t_1-s)} B B^T e^{A^T(t_1-s)} ds\, a$$

$$= \int_{t_0}^{t_1} \left(I + A(t_1-s) + \cdots + \frac{A^k}{k!}(t_1-s)^k + \cdots \right) B B^T e^{A^T(t_1-s)} a\, ds$$

$$= B \int_{t_0}^{t_1} B^T e^{A^T(t_1-s)} a\, ds + AB \int_{t_0}^{t_1} (t_1-s) B^T e^{A^T(t_1-s)} a\, ds$$

$$+ \cdots + A^k B \int_{t_0}^{t_1} \frac{(t_1-s)^k}{k!} B^T e^{A^T(t_1-s)} a\, ds + \cdots$$

$$\Rightarrow \quad d \in \operatorname{Im} [B \;\; AB \; \cdots \; A^{n-1} B \; \cdots]$$

$$= \operatorname{Im} [B\; AB \; \cdots \; A^{n-1} B] = \operatorname{Im} \Gamma$$

$$\Rightarrow \quad \operatorname{Im} W = \operatorname{Im} \Gamma \quad !$$