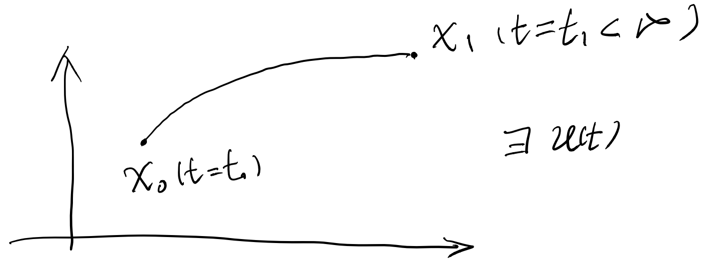
Reachability

Consider

$$\dot{x} = A(t)x + B(t)u \qquad (\dot{x} = f(x,u,t))$$

$$x(t_0) = x_0, \qquad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m$$

$n \geq m$



Controllable: if $\forall\, x_0 \in \mathbb{R}^n$, and $\forall\, x_1 \in \mathbb{R}^n$,
$\exists\, u(t)$, s.t. $x(t_1) = x_1$, $t_1 < \infty$.

Reachable: if $x_0 = 0$

Null-Controllable: if $x_1 = 0$

Recall:

$$x(t) = \Phi(t,t_0)x(t_0) + \int_{t_0}^{t} \Phi(t,s)B(s)u(s)\,ds$$

$$x_1 \overset{?}{=} \Phi(t_1,t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1,s)B(s)\underset{?}{u(s)}\,ds$$

$$\int_{t_0}^{t_1} \Phi(t_1,s)B(s)u(s)\,ds = d := x_1 - \Phi(t_1,t_0)x_0$$

Controllable $\Rightarrow$ $\exists\, u(t)$ $\forall\, d \in \mathbb{R}^n$

$$\Rightarrow \quad \int_{t_0}^{t_1} \Phi(t_1,s)B(s)u(s)\,ds = d$$

solvable in terms of $u(t)$ for all $d \in \mathbb{R}^n$?

We use $\quad L u := \int_{t_0}^{t_1} \underbrace{\Phi(t_1, s)}_{n\times n} \underbrace{B(s)}_{n\times m} u(s)\, ds$

$\Rightarrow \quad Lu = d$

Recall: for $Pu = d$

where P is a constant matrix

$u \in \mathbb{R}^m$, $d \in \mathbb{R}^n$, $\Rightarrow P_{n\times m}$

$\forall d \in \mathbb{R}^n$, $\exists$ solution u?

"onto": iff P has full row rank!

$\Rightarrow \quad \underline{n \leq m}$ $\quad [1111\cdots]$ "fat" matrix

$\quad [\;]$ "skinny" matrix

Let $F(t) = \Phi(t_1, t) B(t)$: $n \times m$ $\quad (n > m)$

Def: $f_1(t), \cdots, f_n(t)$ are said to be linearly independent on time interval $[t_0, t_1]$ if $\sum_{i=1}^{n} c_i f_i(t) = 0$, where $c_i \in \mathbb{R}$

$\forall t \in [t_0, t_1] \Rightarrow c_1 = c_2 = \cdots = c_n = 0$.

Example: ① $1, t, \cdots, t^k$ are linearly independent over any $[t_0, t_1]$

② $\sin t, \sin 2t, \cdots$ $\quad [-\pi, \pi]$

or $\cos t, \cos 2t, \cdots$

Thm: $f_1(t), \cdots, f_n(t)$ are linearly independent on $[t_0, t_1]$ iff

$$W := \int_{t_0}^{t_1} F(s) F^T(s)\, ds \text{ is nonsingular}$$

where $F(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}$

Example: $f_1 = 1, \quad f_2 = t$

$$\int_{t_0}^{t_1} \begin{bmatrix} 1 \\ s \end{bmatrix} \begin{bmatrix} 1 & s \end{bmatrix} ds = \int_{t_0}^{t_1} \begin{bmatrix} 1 & s \\ s & s^2 \end{bmatrix} ds$$

$$= \begin{bmatrix} t_1 - t_0 & \frac{1}{2}(t_1^2 - t_0^2) \\ \frac{1}{2}(t_1^2 - t_0^2) & \frac{1}{3}(t_1^3 - t_0^3) \end{bmatrix}$$

$t_0 = 0 \Rightarrow \det \; \frac{1}{3} t_1^4 - \frac{1}{4} t_1^4 = \frac{t_1^4}{12} \neq 0$ if $t_1 \neq 0$

Proof: "only if": by contradiction, i.e. assuming $f_1(t), \cdots, f_n(t)$ are linearly independent on $[t_0, t_1]$ but W is singular.

W singular $\Rightarrow \exists\, a \neq 0 \in \mathbb{R}^n$, s.t.

$Wa = 0 \quad \Rightarrow a^T W a = 0$

$$\Rightarrow \int_{t_0}^{t_1} a^T F(s) F^T(s) a \, ds = 0 \qquad a = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$\Downarrow$$

$$= \int_{t_0}^{t_1} \| F^T(s) a \|^2 ds = 0$$

$$\Rightarrow \| F^T(t) a \|^2 = 0 \quad \forall \, t \in [t_0, t_1]$$

$$\Rightarrow F^T(t) a = 0 \quad \forall \, t \in [t_0, t_1]$$

$$\Rightarrow a^T F(t) = 0$$

$$\Rightarrow \sum_{i=1}^{n} a_i f_i(t) = 0 \qquad \forall \, t \in [t_0, t_1]$$

$\Rightarrow$ $f_1(t), \cdots, f_n(t)$ are not linearly independent ✗.

Sufficiency: for you to show.

Remark: we can see $a^T W a \geq 0$

If W is nonsingular $\Rightarrow a^T W a > 0$ if $a \neq 0$.

$\Rightarrow$ W is positive definite

if $f_1, \cdots, f_n$ are linearly independent.

Now consider

$$(*) \quad \int_{t_0}^{t_1} F(s) u(s) ds = d, \quad d \in \mathbb{R}^n, u \in \mathbb{R}^m$$

Thm: $\exists\, u(t)$ that solve (*) $\forall\, d \in \mathbb{R}^n$

iff $W(t_0, t_1) = \int_{t_0}^{t_1} F(s) F^T(s)\, ds$ is nonsingular

Proof: "if": $\forall\, d \in \mathbb{R}^n$, let

$$u(t) = F^T(t)\, W^{-1}(t_0, t_1)\, d$$

$$\Rightarrow \quad \int_{t_0}^{t_1} F(s) F^T(s) W^{-1}(t_0, t_1)\, d\, ds = d$$

"only if": assume $W(t_0, t_1)$ is singular, $\Rightarrow \exists\, a \neq 0 \in \mathbb{R}^n$, s.t.

$$W(t_0, t_1)\, a = 0$$

$$\Rightarrow \quad \underline{a^T F(t) = 0 \quad \forall t \in [t_0, t_1]}$$

then $\int_{t_0}^{t_1} F(s) u(s)\, ds = a \Rightarrow$

$$\int_{t_0}^{t_1} a^T F(s) u(s)\, ds = a^T a$$

$$\Rightarrow \quad 0 = a^T a = \|a\|^2 \neq 0 \quad \times\!\!\times.$$

Now let's get back to

$$(**) \quad \int_{t_0}^{t_1} \underbrace{\Phi(t_1, s) B(s)}_{F(s)} u(s) ds = d \,, \quad \forall d \in \mathbb{R}^n$$

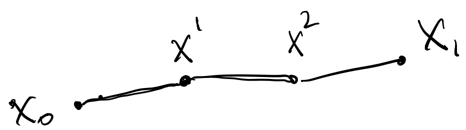
Thm: 1) (**) is solvable $\forall d \in \mathbb{R}^n$, or the system is reachable (controllable)

iff

$$W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_1, s) B(s) B^T(s) \Phi^T(t_1, s) ds$$

is nonsingular — Reachability Gramian

and $u^*(t) = B^T(t) \Phi^T(t_1, t) W^{-1}(t_0, t_1) d$ is a solution, where $d = x_1 - \Phi(t_1, t_0) x_0$.



2) If $W(t_0, t_1)$ is singular, then

$$\int_{t_0}^{t_1} \Phi(t_1, s) B(s) u(s) ds = d$$

is solvable iff $d \in \operatorname{Im} W(t_0, t_1)$

(i.e. $\exists\, a$ s.t. $d = W(t_0, t_1) a$),

and $u^*(t) = B^T(t) \Phi^T(t_1, t) a$

solves the equation.

Plug in u^* we have

$$\int_{t_1}^{t_1} \underbrace{\Phi(t_1, s) B(s) B^T(s) \Phi^T(t_1, s)} a \, ds$$

$$= W(t_0, t_1) a = d \quad !$$