Linear dynamical systems (Continued)

Recap:

For

$$\dot{x} = A(t)x$$
$$x(t_0) = a$$

- Cauchy problem

The solution is $x(t) = \underset{n\times n}{\Phi}(t, t_0)a$

Where $\Phi(t, t_0)$ is the so called state transition matrix

$$\boxed{\begin{aligned} &\dot{\Phi}(t, t_0) = A(t)\Phi(t, t_0) \\ &\Phi(t_0, t_0) = I \\ &\text{And } \Phi(t, t_0) \text{ is nonsingular } \forall t, t_0 \end{aligned}}$$

$$\Phi(t, s) = \Psi(t)\Psi^{-1}(s)$$

where $\Psi(t)$ is a fundamental matrix

Then by letting $z(t) = \Phi(t_0, t)x(t)$

we obtained the solution to

$$\dot{x} = A(t)x + B(t)u(t)$$

as $\boxed{x(t) = \Phi(t, t_0)a + \int_{t_0}^{t} \Phi(t, s)B(s)u(s)ds}$

Now adding in $y(t) = C(t)x(t) + D(t)u(t)$

and setting $a=0$

$$\Rightarrow \quad y(t) = \int_{t_0}^{t} C(t)\Phi(t,s)B(s)u(s)ds + D(t)u(t)$$

Recall in input-output description:

$$y(t) = \int_{t_0}^{t} G(t,s)u(s)ds + D(t)u(t)$$

$$\Rightarrow \quad \boxed{G(t,s) = C(t)\Phi(t,s)B(s)}$$

Recall that the system is finite dimensional

if $\quad G(t,s) = H(t)K(s)$

Since $\quad \Phi(t,s) = \Psi(t)\Psi^{-1}(s)$

$$\Rightarrow \quad C(t)\Phi(t,s)B(s) = \underbrace{C(t)\Psi(t)}_{H}\,\underbrace{\Psi^{-1}(s)B(s)}_{K}$$

time-invariant:

$$G(t,s) = G(t-s)$$

$$\Rightarrow \quad B(t) = B, \quad C(t) = C$$

$$\Rightarrow \quad G(t,s) = C\Phi(t,s)B \stackrel{?}{=} G(t-s)$$

$$\Rightarrow \quad \Phi(t,s) = \Phi(t-s)$$

Claim: $\boxed{\begin{array}{l}\Phi(t,s) = \Phi(t-s) \text{ iff} \\ A(t) = A \text{ a constant matrix}\end{array}}$

Remark: from $G(t-s) = C\,\Phi(t-s)B$

We can construct $\bar{G}(t-s) = \underbrace{C e^{at}}_{\bar{C}(t)} \Phi(t-s) \underbrace{e^{-as} B}_{\bar{B}(t)}$

$$\bar{G}(t-s) = C \underbrace{e^{a(t-s)} \Phi(t-s)}B \quad - \text{time-invariant}$$

How do we calculate $\Phi(t,s)$?

Example: $\dot{x} = a(t)x \qquad x \in \mathbb{R}$

$$\int_{t_0}^{t} \frac{dx}{x} = \int_{t_0}^{t} a(s)\,ds$$

$\ln \frac{x(t)}{x(t_0)} \Leftarrow$

$$\ln x(t) - \ln x(t_0) = \int_{t_0}^{t} a(s)\,ds$$

$$\Rightarrow \quad x(t) = e^{\int_{t_0}^{t} a(s)\,ds}\, x(t_0)$$

$$\Rightarrow \quad \Phi(t,t_0) = e^{\int_{t_0}^{t} a(s)\,ds} = \varphi(t)\varphi^{-1}(t_0)$$

$$= e^{\left(\int_0^t a(s)\,ds + \int_{t_0}^{0} a(s)\,ds\right)}$$

$$= e^{\int_0^t a(s)\,ds}\, e^{-\int_0^{t_0} a(s)\,ds}$$

$$\varphi(s) := e^{\int_0^t a(s)d}, \quad \varphi^{-1}(t_0) = e^{-\int_0^{t_0} a(s)\,ds}$$

Now let $a(t) = a$

$$\Rightarrow \quad \Phi(t,t_0) = e^{a(t-t_0)} = \Phi(t-t_0)$$

Bad news: for a general

$$\dot{x} = A(t)x$$

it is almost never possible to find the explicit expression of $\Phi(t,s)$.

Time-invariant systems

$$\dot{x} = Ax + Bu \quad , \quad x \in \mathbb{R}^n$$

$$y = Cx + Du \quad \quad u \in \mathbb{R}^m, \quad y \in \mathbb{R}^p$$

We focus on the state transition matrix for $\dot{x} = Ax$ (for A)

Recall that for the scalar case

$$\dot{x} = ax$$

we have $\Phi(t,s) = \Phi(t-s) = e^{a(t-s)}$.

Since $e^{at} = 1 + at + \cdots + \frac{1}{k!}(at)^k + \cdots \quad t \geq 0$

Define

$$\boxed{e^{At} := I_{n\times n} + At + \cdots + \frac{1}{k!}A^k t^k + \cdots + \cdots}$$

We need to check $\|e^{At}\| < \infty$

$$\Rightarrow \quad \|e^{At}\| = \|I_{n\times n} + At + \cdots + \frac{1}{k!}A^k t^k + \cdots\|$$

$$\leq 1 + \|A\| t + \cdots + \frac{1}{k!}\|A\|^k t^k + \cdots$$

$$= e^{\|A\| t} < \infty$$

$\Rightarrow$ e^{At} exists!

Properties of e^{At}

1. if $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$, then $e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}$.

2. $e^{P^{-1}APt} = P^{-1} e^{At} P$

3. If $A_1 A_2 = A_2 A_1$, then $e^{(A_1 + A_2)t} = e^{A_1 t} e^{A_2 t}$

4. e^{At} is always nonsingular and $(e^{At})^{-1} = e^{-At}$

$\Rightarrow$ $e^{At} e^{-At} = I$

Proof: since $A(-A) = (-A)A = -A^2$

from 3) we have $e^{(A-A)t} = e^{At} e^{-At} = I$

done!

5. $\dfrac{d e^{At}}{dt} = A e^{At}$ $\Rightarrow$ e^{At} is a fundamental matrix for $\dot{x} = Ax$

$$\Phi(t, s) = \psi(t)\psi^{-1}(s) = e^{At} e^{-As} = e^{A(t-s)}$$

$\Rightarrow$ $\Phi(t-s) = e^{A(t-s)}$!

For $\dot{x} = Ax$ $\quad (t_0 = 0)$

We can take Laplace transform on both sides: denote $X(s) = \mathcal{L}(x(t))$

$$\Rightarrow \quad sX(s) - x(0) = AX(s)$$

$$(sI - A)X(s) = x(0)$$

$$\Rightarrow \quad X(s) = (sI-A)^{-1}x(0)$$

$$\Rightarrow \quad x(t) = \mathcal{L}^{-1}(X(s)) = \underline{\mathcal{L}^{-1}((sI-A)^{-1})}\,x(0)$$

on the other hand $x(t) = \Phi(t,0)x(0)$

$$= e^{At}x(0)$$

$$\Rightarrow \quad e^{At} = \underline{\mathcal{L}^{-1}((sI-A)^{-1})} \; !$$

Example: point mass

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0]x$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$e^{At} = I_{2\times 2} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t + \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 t^2 + \cdots$$

$$= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad e^{A(t-s)} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad G(t-s) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}}_{H(t)} \underbrace{\begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{K(s)} = t-s$$

Recall $G(t-s) = t-s$