Introduction

- Systems theory vs Control theory
  - overlap in concepts
  - different perspectives $\Rightarrow$ different derivations
  - Rigorous math. derivation $\Rightarrow$ how and why!

- Focus of this lecture
  - what a system (linear system in particular)
  - how to model a linear system
    - input-output description
    - state space

What is a system?

$$u \longrightarrow \boxed{\Sigma} \longrightarrow y$$

u: input
y: output

where $u: T \to \mathbb{R}^m$

$y: T \to \mathbb{R}^p$

T: $\mathbb{Z}$ — disrete time

$\mathbb{R}$ $(\mathbb{R}^+)$ — continuous time system

SISO: if $m = p = 1$,

MIMO: otherwise

Example: point mass

$m\ddot{y} = u$ $\quad m = 1, \quad t_0 = 0$ — initial time

$$\int_0^t \ddot{y}(s)\,ds = \int_0^t u(s)\,ds$$

$$\dot{y}(t) - \dot{y}(0) = \underline{\int_0^t u(s)\,ds}$$

- Setting $\dot{y}(0) = 0$

$$\int_0^t \dot{y}(r)\,dr = \int_0^t \left(\int_0^r u(s)\,ds \right) dr$$

$$\Downarrow$$

$$\Rightarrow \quad y(t) - y(0) = \int_0^t (t-s)\,u(s)\,ds$$

- Setting $y(0) = 0$

$$\underline{y(t) = \int_0^t (t-s)\,u(s)\,ds}$$

- Memoryless system: if $y(t)$ depends only on $u(t)$.

otherwise it is a system with memory (our focus)

- relaxed: if $u(t) \equiv 0 \Rightarrow y(t) \equiv 0$

Linear system:

Notation: $y(t) = f_\Sigma(u)$

linear: if $f_\Sigma(\alpha u_1 + \beta u_2)$

$$= \alpha f_\Sigma(u_1) + \beta f_\Sigma(u_2)$$

provided the system is relaxed.

Input-output description (of linear systems)

$$y(t) = \int_{t_0}^{t} G(t,s)\, u(s)\, ds + D(t)\, u(t)$$

$G(t,s)$: impulse response.

point mass: $D(t) \equiv 0$, $G(t,s) = (t-s)$

t_0 can be taken as 0, $-\infty$, or any $t_0 \in \mathbb{R}$.

Time-invariant systems:

Define: $u_T(t) = \begin{cases} u(t-T) & t \geq t_0 + T \\ 0 & \text{otherwise} \end{cases}$

a system is time-invariant if

$$f_\Sigma(u_T) = y_T \qquad \forall T > 0$$

$$y_T(t) = \int_{t_0}^{t-T} G(t-T, s)\, u(s)\, ds + D(t-T)\, u(t-T)$$

$$f_\Sigma(u_T) = \int_{t_0}^{t} G(t, s)\, u_T(s)\, ds + D(t)\, u(t-T)$$

$$\overset{s=r+T}{=} \int_{t_0-T}^{t-T} G(t, r+T)\, u_T(r+T)\, dr + D(t)\, u(t-T)$$

$$= \int_{t_0}^{t-T} G(t, r+T)\, u(r)\, dr + D(t)\, u(t-T)$$

$\Rightarrow$ 1. $D(t-T) = D(t) \quad \forall T > 0 \Rightarrow D(t) = D$

2. $G(t-T, s) = G(t, s+T)$

$\Rightarrow \boxed{G(t, s) = G(t-s)}$ - Time-invariant

Finite dimensional systems:

$$\Rightarrow \quad \underset{p\times m}{G(t,s)} = \underset{p\times n}{H(t)}\, \underset{n\times m}{K(s)}$$

Example: $\underline{e^{a(t-s)} = e^{at} \cdot e^{-as}}$

State space modelling:

Introduce:

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \qquad x \in \mathbb{R}^n$$

— state variable.

a linear system:

$$\dot{x} = A(t)\,x(t) + B(t)\,u(t)$$

$$y(t) = C(t)\,x(t) + D(t)\,u(t)$$

$x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$.

$\underline{A_{n\times n},\ B_{n\times m},\ C_{p\times n},\ D_{p\times m}}$

Example: point mass

$$\ddot{y} = u$$

$$x_1(t) = y(t)$$

$$x_2(t) = \dot{y}(t)$$

$$\dot{x}_1 = \dot{y}(t) = x_2$$

$$\dot{x}_2 = \ddot{y}(t) = u$$

$$y = x_1$$

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases}$$