The stability and instability of partial realizations

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A finite or infinite sequence of real numbers is said to be stable if it admits minimal realization by a stable linear system. It is shown that the preservation of stability as such a sequence is truncated or extended is not a generic property even among stable sequences. This is of interest for identification of linear systems from partial data and for partial realization of random systems, in which cases it constitutes a negative result. Certain finite sequences have infinitely many minimal realizations each having different stability properties. In this case, a graphical criterion in the spirit of the Nyquist criterion is derived to exploit this lack of uniqueness in order to determine whether one can achieve a stable pole placement by a judicious choice of partial realization.

Keywords: Stability, Partial realizations, Generic properties, Linear systems, Nyquist criterion.

1. Introduction

Given a finite sequence

$\gamma := (\gamma_1, \gamma_2, \ldots, \gamma_N)$

of real numbers, consider triplets of matrices $\Sigma := (A, B, C)$ with the property that

$$CA^{i-1}B = \gamma_i, \quad i = 1,2,\ldots,N. \tag{1}$$

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This requires that $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$ and $C \in \mathbb{R}^{1\times n}$ for some arbitrary nonnegative integer n. Such a triplet is called a *realization* of γ, and the number n is its *dimension*. Representations of this type play a central role in systems theory [6–9,13,15,16,19,20]. It is easy to see that any finite sequence γ has a realization. Let $\delta(\gamma)$ be the unique integer with the property that γ has a realization of dimension $\delta(\gamma)$, but none of smaller dimension; $\delta(\gamma)$ is the *McMillan degree* of γ. A realization of dimension $\delta(\gamma)$ is said to be *minimal*. A *partial realization* of a (finite or infinite) sequence γ is a realization of some subsequence of γ of type $(\gamma_1, \gamma_2, \ldots, \gamma_M)$.

To each realization (A,B,C) of

$\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_N)$

we associate the rational function

$$W(z) = C(zI - A)^{-1}B \tag{2}$$

called the *transfer function*. In view of (1), the N first coefficients of its Laurent series about $z = \infty$,

$$W(z) = \sum_{i=1}^{\infty} CA^{i-1}Bz^{-i}, \tag{3}$$

are precisely the components of the sequence γ. If W has all its poles in the open left half plane $\mathbb{C}^-$, the realization (A,B,C) is said to be *stable*; if they are all in the open right half plane $\mathbb{C}^+$, (A,B,C) is *completely unstable*. (All results of this paper remain valid with $\mathbb{C}^-$ and $\mathbb{C}^+$ replaced in this definition by the open regions inside and outside the unit circle respectively.) The sequence γ will be called *stable* if it has a stable minimal realization. Note that, as we shall see below, a stable sequence may well have an unstable minimal realization also.

We shall represent the sequence γ as a point in the space $\mathbb{R}^N$ equipped with the usual (Euclidean) topology. The following theorem is our main result.

Theorem 1. *Let $N \geqslant 3$, and let M be the largest even number smaller than N. Then there is a nonempty open set $U \subset \mathbb{R}^N$ of stable sequences $(\gamma_1, \gamma_2, \ldots, \gamma_N)$ with the property that all minimal realizations of the partial sequence $(\gamma_1, \gamma_2, \ldots, \gamma_M)$ are completely unstable.*

This statement, which was motivated by a question raised by Kalman in [8], has the following interpretations and consequences. We denote by $\mathcal{D}_N \subset \mathbb{R}^N$ the open set of stable sequences $(\gamma_1, \ldots, \gamma_N)$.

Corollary 1. *Let $N \geqslant 3$. Then there exists no open dense subset in D_N of stable sequences $(\gamma_1, \gamma_2, \ldots, \gamma_N)$ with the property that all partial sequences*

$$\{(\gamma_1, \gamma_2, \ldots, \gamma_k); k < N\}$$

are stable.

Consequently the preservation of stability of partial realizations is *not* a generic property among stable sequences, as one may have hoped [8; bottom of p. 23]. (See Definition 2 and the preceding paragraphs in Section 2 for the definition of 'generic'.)

Corollary 2. *Let M be any natural number, and let N be the smallest even number greater than $M+1$. Then there is a nonempty open set $U \subset \mathbb{R}^N$ of sequences $\gamma := (\gamma_1, \gamma_2, \ldots, \gamma_N)$ such that the partial sequence $(\gamma_1, \gamma_2, \ldots, \gamma_M)$ is stable but all minimal realizations of γ are completely unstable.*

This has consequences for the identification problem. In fact, it follows from Corollary 2 that we cannot infer stability of a system from the stability of its partial realizations, even in the generic case, answering in the negative a question also raised in [8] by Kalman (p. 25).

Finally, let us consider the algorithms for partial realization of random systems proposed in [13,14] and in [5]. The property of the covariance sequence needed for these procedures to work is *not* generically satisfied among stable sequences, since it requires that a partial sequence of a stable sequence is stable.

2. Preliminaries

To each sequence $\gamma := (\gamma_1, \gamma_2, \ldots, \gamma_N)$ of real numbers we associate the family $H(\gamma)$ of all Hankel matrices

$$H_{ij} = \begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_j \\ \gamma_2 & \gamma_3 & \cdots & \gamma_{j+1} \\ \vdots & & & \\ \gamma_i & \gamma_{i+1} & \cdots & \gamma_{i+j-1} \end{bmatrix} \tag{4}$$

such that $i + j - 1 \leqslant N$. The connection between the ranks of such matrices and the McMillan degrees of the corresponding partial sequences is well known and thoroughly studied in the literature [6,7,16,19]. The particular formulation given by the following theorem appears in [5], but at least part (i) is easily obtained from Theorem 11 in [16].

Theorem 2. *Given a sequence*

$$\gamma := (\gamma_1, \gamma_2, \ldots, \gamma_N)$$

of real numbers, let

$$\nu := \{\nu(0), \nu(1), \ldots, \nu(n)\}$$

be a sequence of integers defined in the following way. Set $\nu(0) := 0$, and, for $k = 0,1,2,\ldots$ until the process stops, define $\nu(k+1)$ to be the smallest $j = \nu(k)+1, \nu(k)+2, \ldots, N-\nu(k)$, if any, such that $H_{\nu(k)+1,j}$ has full rank. Then

(i) *For $M = 1,2,\ldots,N$, the partial sequence $\{\gamma_1, \gamma_2, \ldots, \gamma_M\}$ has McMillan degree $\nu(k)$ if and only if*

$$\nu(k-1) + \nu(k) \leqslant M < \nu(k) + \nu(k+1) \tag{5}$$

where $\nu(-1) := 0$ and $\nu(n+1) := N+1$.

(ii) *A square Hankel matrix $H_{ii} \in H(\gamma)$ is nonsingular if and only if $i \in \nu$.*

Therefore it is reasonable to call

$$\{\nu(0), \nu(1), \ldots, \nu(n)\}$$

the *degree indices* of γ. A family

$$\{\Sigma_0, \Sigma_1, \ldots, \Sigma_n\}$$

of partial realizations of γ with the property that, for $k = 0,1,\ldots,n$, Σ_k is a minimal realization of $\{\gamma_1, \gamma_2, \ldots, \gamma_M\}$ for each M satisfying (5) is called a *complete family of minimal partial realizations* of γ [5]. Clearly, Σ_k has dimension $\nu(k)$, and $\Sigma_0 = (0,0,0)$.

Now let $\Gamma_0(z)$ be an arbitrary formal power series of type

$$\Gamma_0(z) = \sum_{i=1}^{\infty} \gamma_i z^{-i} \tag{6}$$

and consider the following algorithm proposed by Magnus [12]. For $k = 1,2,3,\ldots$ while $\Gamma_{k-1} \neq 0$, invert Γ_{k-1}, normalize by multiplying with a real number $\beta_{k-1} \in 0$, and split

$$\beta_{k-1}/\Gamma_{k-1}(z) = \alpha_k(z) - \Gamma_k(z) \tag{7}$$

into a polynomial (principal) part

$$\alpha_k(z) = z^{d(k)} - \alpha_{k1} z^{d(k)-1} - \cdots - \alpha_{k,d(k)} \tag{8}$$

and a formal power series Γ_k of type (6). In each step, β_{k-1} is chosen so that α_k is monic. This process terminates in a finite number of steps if and only if Γ_0 converges to a rational function P/Q in the neighborhood of $z = \infty$, in which case it is equivalent to the Euclidean algorithm applied to the pair (P, Q) of relatively prime polynomials.

Theorem 3 (Magnus). *Let*

$$\gamma := (\gamma_1, \gamma_2, \ldots, \gamma_N)$$

be a sequence with degree indices

$$\{\nu(0), \nu(1), \ldots, \nu(n)\}$$

and let

$$\{\Sigma_0, \Sigma_1, \ldots, \Sigma_n\}$$

be a complete family of minimal partial realizations of γ. *Let* Γ_0 *be the transfer function of* Σ_n. *Then the algorithm* (7) *terminates in n steps, and* $\nu(k+1) = \nu(k) + d(k)$ *for* $k = 0,1,2,\ldots,n-1$; $\nu(0) = 0$. *Also, for each* $k = 0,1,2,\ldots,n$, *the transfer function of* Σ_k *is* $W_k = P_k/Q_k$ *where* P_k *and* Q_k *are relatively prime polynomials generated by the three-term recursions*

$$P_{k+1}(z) = \alpha_{k+1}(z) P_k(z) - \beta_k P_{k-1}(z); \quad P_{-1} = -1, \quad P_0 = 0; \tag{9a}$$

$$Q_{k+1}(z) = \alpha_{k+1}(z) Q_k(z) - \beta_k Q_{k-1}(z); \quad Q_{-1} = 0, \quad Q_0 = 1. \tag{9b}$$

Moreover, for $k = 1,2,\ldots,n-1$, P_k *and* P_{k+1} *are relatively prime as are* Q_k *and* Q_{k+l}.

This interpretation of Magnus' results [12] (which were not developed in the context of realization theory) can be found in [5]; also see [10].

As in [5] we define the *parameter sequence*

$$\rho := (\rho_1, \rho_2, \ldots, \rho_N)$$

of γ in the following way. For $k = 0,1,2,\ldots,$ set

$$\rho_{2\nu(k-1)+i} = \begin{cases} 0 & \text{if } 0 \leq i < d(k), \\ \beta_{k-1} \neq 0 & \text{if } i = d(k), \\ \alpha_{k,i-d(k)} & \text{if } d(k) < i \leq 2d(k), \end{cases} \tag{10}$$

until all components of ρ have been determined. The following result is proved in [5].

Theorem 4. [5]. *Let*

$$\Phi_N : \mathbb{R}^N \to \mathbb{R}^N$$

be the function sending each sequence

$$\gamma := (\gamma_1, \gamma_2, \ldots, \gamma_N)$$

to its parameter sequence ρ. *Then* Φ_N *is a bijection, and, for* $M < N$, $\hat{\gamma} \in \mathbb{R}^M$ *is a partial sequence of* γ *if and only if* $\hat{\rho} := \Phi_M(\hat{\gamma})$ *is a partial sequence of* $\rho := \Phi_N(\gamma)$. *Moreover, if* γ *has degree indices* $\{\nu(0), \nu(1), \ldots, \nu(n)\}$, *the transfer functions* $(W_0, W_1, \ldots, W_{n-1})$ *are uniquely determined by* γ, *whereas* W_n *is unique if and only if* $N \geq 2\nu(n)$. *If* $N < 2\nu(n)$, *the last* $2\nu(n) - N$ α*-parameters needed for determining* W_n *are arbitrary.*

A property of points in the Euclidean space $\mathbb{R}^N$ is said to be *generic* if all $x \in \mathbb{R}^N$ have this property except perhaps for those x which are contained in some proper algebraic subset

$$X = \{x \mid f_i(x) = 0; \; f_i \text{ polynomial}; \; i = 1,2,\ldots,p\} \tag{11}$$

of $\mathbb{R}^N$. Note that the set of points which enjoy a generic property contains an open dense subset of $\mathbb{R}^N$. In fact, a proper algebraic subset X is a closed subset with no interior. To see this, observe that, if X had an interior point x_0, it would contain an ϵ-ball centered about x_0. Then, for each defining equation $f_i(x) = 0$, expand f_i in a Taylor series about x_0 to see that this would require the derivatives of f_i of all orders to vanish at $x = x_0$. Consequently all polynomials f_i must be identically zero, and hence $X = \mathbb{R}^N$, contradicting the assumption that X be a proper subset of $\mathbb{R}^N$.

Let us consider two examples of generic properties.

Example 1. Let $(\rho_1, \rho_2, \ldots, \rho_{2n}) \in \mathbb{R}^{2n}$ have the property that $\rho_{2i+1} \neq 0$ for $i = 0,1,\ldots,n-1$. This is a generic property since the exceptional set is

$$X = \left\{\rho \mid \prod_{i=0}^{n-1} \rho_{2i+1} = 0\right\}.$$

For later reference, define $V := \mathbb{R}^{2n} - X$ to be the set of points having this generic property.

Example 2. The property of $(\gamma_1, \gamma_2, \ldots, \gamma_{2n}) \in \mathbb{R}^{2n}$ that all square Hankel matrices $H_{ii} \in H(\gamma)$ are nonsingular is a generic property, as was pointed out by Brockett [1]. In fact, the exceptional set is

$$X = \{\gamma \mid \det H_{ii} = 0; i = 1,2,\ldots,n\}.$$

Let W denote the set of points in $\mathbb{R}^{2n}$ enjoying this generic property.

More generally:

Definition 1. Suppose $S \subset \mathbb{R}^N$ is a subset. We shall say that a property of points $x \in S$ is *generic* if all $x \in S$ have this property except perhaps for those x which are contained in the proper intersection $X \cap S \subset S$ of S with an algebraic subset $X \subset \mathbb{R}^N$.

Thus, if $S \subset \mathbb{R}^2$ is the open left-half plane then the property of points $(x,y) \in S$ that $y \neq 0$ is a generic property.

By Theorem 2, each $\gamma \in W$ has degree indices $\{0,1,2,\ldots,n\}$, and therefore it has a parameter sequence $\rho \in V$. In fact, *a fortiori*, there is a one-one correspondence between V and W (Theorem 4). Now the monic polynomials $\alpha_1, \alpha_2, \ldots, \alpha_n$ are linear, and P_k and Q_k have degrees $k-1$ and k respectively, and therefore (9) reads

$$\begin{cases} P_{k+1}(z) = (z - \rho_{2k+2})P_k(z) \\ \qquad - \rho_{2k+1}P_{k-1}(z); \\ P_{-1} = -1, \quad P_0 = 0; & \text{(12a)} \\ Q_{k+1}(z) = (z - \rho_{2k+2})Q_k(z) \\ \qquad - \rho_{2k+1}Q_{k-1}(z); \\ Q_{-1} = 0, \quad Q_0 = 1. & \text{(12b)} \end{cases}$$

Since $N := 2n = 2\nu(n)$, the transfer function W_n of γ is uniquely determined by γ, whereas the partial sequence $(\gamma_1, \gamma, \ldots, \gamma_{2n-1})$ has a one-parameter family of transfer functions, ρ_{2n} being the arbitrary parameter (Theorem 4).

The recursions (12) have a longer history than their generalization (9), going back all the way to Chebyshev [2] and Stieltjes [17,18]. They provide a well-known method for tridiagonalization of matrices and computation of eigenvalues, introduced by Lanczos [11]. Therefore $\{Q_0, Q_1, \ldots, Q_n\}$ and $\{P_0, P_1, \ldots, P_n\}$ are usually referred to as the *Lanczos polynomials of the first and second kind* respectively [4].

3. Proof of Theorem 1

Lemma 1. *The function $\phi: W \to V$ obtained by restricting the domain and the range of the function Φ_{2n} (defined in Theorem 4) is a homeomorphism.*

Proof. By definition, $\gamma \in W$ holds if and only if $\det H_{ii} \neq 0$ for $i = 1,2,\ldots,n$. But this happens if and only if γ has degree indices $\{0,1,2,\ldots,n\}$ (Theorem 2), which is equivalent to $\rho \in V$. Hence $\Phi_{2n}(W) = V$. Now Φ_{2n} is a bijection (Theorem 4), thus so is ϕ. Let $\phi_1 : V \to \mathbb{R}^{2n}$ be the function, defined via (12), sending ρ to the coefficients of P_n and Q_n, and let $\phi_2 : \phi_1(V) \to \mathbb{R}^{2n}$ be the function assigning to the coefficients of P_n and Q_n the coefficients of the powers $z^{-1}, z^{-2}, \ldots, z^{-2n}$ in the Laurent series about $z = \infty$ of $W_n := P_n/Q_n$; note that P_n and Q_n are relatively prime (Theorem 3). Both ϕ_1 and ϕ_2 are polynomials and hence continuous. But

$$\gamma = \phi^{-1}(\rho) = (\phi_2 \circ \phi_1)(\rho)$$

for all $\rho \in V$, and therefore $\phi^{-1} : V \to W$ is a continuous bijection. Since V is locally compact and W is Hausdorff ϕ^{-1}, and hence ϕ, is a homeomorphism. □

Let R_0 and R_1 be arbitrary real monic polynomials of degrees n and $n-1$ respectively and let λ be an arbitrary real number. Then the triplet (R_0, R_1, λ) contains $2n$ arbitrary real parameters and can therefore be represented as a point $x \in \mathbb{R}^{2n}$. The polynomials R_0 and R_1 are relatively prime if and only if the Euclidean algorithm applied to the pair (R_0, R_1), i.e.

$$\begin{aligned} R_0(z) &= \pi_1(z)R_1(z) + R_2(z), \\ &\quad \deg R_2 < \deg R_1, \\ R_1(z) &= \pi_2(z)R_2(z) + R_3(z), \\ &\quad \deg R_3 < \deg R_2, \\ &\dots \\ R_{m-2}(z) &= \pi_{m-1}(z)R_{m-1}(z) + R_m(z), \\ &\quad \deg R_m < \deg R_{m-1}, \\ R_{m-1}(z) &= \pi_m(z)R_m(z), \end{aligned} \tag{13}$$

produces a final polynomial R_m of degree zero. Here $m \leqslant n$. The algorithm (13) terminates in n steps ($m = n$) if and only if all polynomials $\pi_1, \pi_2, \dots, \pi_n$ are of degree one. In this case R_0 and R_1 are always relatively prime.

Lemma 2. *Satisfying the two conditions*

(i) *Euclidean algorithm* (13) *terminates in n steps* ($m = n$),

(ii) $\lambda \neq 0$,

is a generic property of the point $x \in \mathbb{R}^{2n}$ *corresponding to the triplet* (R_0, R_1, λ).

Proof. Identifying coefficients of powers of z in the first equation of (13), it is easy to see that π_1 is monic and linear and that the coefficients of $R_2(z)$ are polynomials in x. Let $\theta_1(x)$ be the coefficient of z^{n-2} in $R_2(z)$. The set

$$X_1 := \{x \mid \theta_1(x) = 0\}$$

belongs to the exceptional set X of points in $x \in \mathbb{R}^{2n}$ for which the property does not hold. If $x \notin X_1$, continue to the second equation of (13), where then π_2 is linear. It is no restriction to replace $R_1(z)$ by $\theta_1(x)R_1(z)$ there, making π_2 monic; this is just a normalization since $\theta_1(z)$ is a nonzero constant. Let $\theta_2(x)$ be the coefficient of z^{n-3} in $R_3(z)$. As before θ_2 is a polynomial, and

$$X_2 := \{x \mid \theta_2(x) = 0\} \subset X.$$

Moreover, θ_1 is a factor in θ_2, so $X_1 \subset X_2$. Proceeding in this fashion leads in $n-1$ steps to a remainder $R_n(z)$ of degree zero. In fact, $R_n(z) = \theta_{n-1}(x)$, where θ_{n-1} is a polynomial having $\theta_1, \theta_2, \dots, \theta_{n-2}$ as factors. Consequently

$$X_{n-1} := \{x \mid \theta_{n-1}(x) = 0\}$$

consists precisely of those $x \in \mathbb{R}^{2n}$ which correspond to polynomials R_0 and R_1 for which the Euclidean algorithm (13) terminates with $m < n$. Then the exceptional set is

$$X = \{x \mid x_{2n}\theta_{n-1}(x) = 0\},$$

where $x_{2n} = \lambda$. This is clearly a set of type (11). □

Lemma 3. *Let* $Z \subset \mathbb{R}^{2n}$ *be the set of points enjoying the generic property of Lemma 2, and let* $\psi: V \to Z$ *be the function such that* $\psi(\rho)$ *is the* $\mathbb{R}^{2n}$ *representation of* (Q_n, Q_{n-1}, ρ_1) *where* Q_n *and* Q_{n-1} *are given by* (12b). *Then* ψ *is a homeomorphism, and* Z *is open and dense in* $\mathbb{R}^{2n}$.

Proof. In the generic case ($m = n$) the equations of (13) may be identified with (12b) if we take $k = n-1, n-2, \dots, 1$. In fact, modulo a trivial normalization, $R_0, R_1, \dots, R_n$ correspond to $Q_n, Q_{n-1}, \dots, Q_0$ respectively. Hence ψ is a bijection, the Euclidean algorithm (13) being the inverse ψ^{-1}. Consequently, since ψ is continuous, it is a homeomorphism (Brouwer's theorem on invariance of domain). Since the exceptional set X in the proof of Lemma 2 is closed and has no interior (see remark in Section 2), Z is open and dense. □

Note that, in Lemmas 1 and 3, it is necessary to restrict the functions to the generic sets in order to insure that they are homeomorphisms. In fact, allowing the parameters to vary so that the α-polynomials in (9) change degrees will destroy continuity.

We now turn to the proof of Theorem 1. First, for $n \geqslant 2$, take $N = 2n$. Consider all triplets (Q_n, Q_{n-1}, ρ_1) such that Q_n is a stable (all zeros in $\mathbb{C}^-$) monic polynomial of degree n, Q_{n-1} is a completely unstable (all zeros in $\mathbb{C}^+$) monic polynomial of degree $n-1$, and ρ_1 is a nonzero real number. Let Y be the corresponding subset of $\mathbb{R}^{2n}$. It follows from the Routh–Hurwitz theory [3] that Y is a nonempty open set. Therefore, since Z is open and dense (Lemma 3), $Y \cap Z$ is open and nonempty, and consequently so is

$$U := (\psi \circ \phi)^{-1}(Y \cap Z)$$

(Lemmas 1 and 3). Clearly $U \subset W$. Therefore each $\gamma \in U$ has degree indices $\{0,1,2,\dots,n\}$ and a parameter sequence $\rho \in V$. By Theorem 4, the transfer functions W_{n-1} and W_n are unique. They are

$$W_{n-1} = P_{n-1}/Q_{n-1} \quad \text{and} \quad W_n = P_n/Q_n$$

where P_{n-1} and P_n are determined by (12a). Since W_n is stable, γ is stable, whereas W_{n-1} is completely unstable. This establishes the theorem for even N, since all minimal realizations of $(\gamma_1, \gamma_2, \ldots, \gamma_{2n-2})$ have the transfer function W_{n-1}. Next, for $n \geqslant 2$, take $N = 2n-1$. Then merely imbed γ in the extended sequence (γ, γ_{2n}) and proceed as above. Since (γ, γ_{2n}) is stable, so is γ. □

The proof of Corollary 2 is analogous.

4. A graphical criterion for stability of certain partial realizations

Consider a generic sequence

$$(\gamma_1, \gamma_2, \ldots, \gamma_{2n-1}) \in \mathbb{R}^{2n-1},$$

i.e. a sequence with the property that det $H_{ii} \neq 0$ for $i = 1,2,\ldots,n$. Such a sequence has McMillan degree n and, as explained in the end of Section 2 above, it has a one-parameter family of transfer functions $W_n(z; \rho_{2n})$.

We ask, then, whether one can choose a value of the parameter ρ_{2n} so that the corresponding realization is stable. This is in contrast to the study of generic sequences of even length, for which there exists a unique minimal realization which may or may not be stable as we have noted above. Explicitly, in this case we can construct a single contour in the complex plane from the data $(\gamma_1, \ldots, \gamma_{2n-1})$ such that the winding number of this contour about the free parameter $1/\rho_{2n}$ determines the stability and the instability of the minimal realization corresponding to ρ_{2n}. Our 'Nyquist Criterion' arises from consideration of the Lanczos polynomials Q_{n-1}, Q_{n-2} which are computed from $\gamma = (\gamma_1, \ldots, \gamma_{2n-1})$ using Chebyshev's recursion formulae (12b). From Q_{n-1}, Q_{n-2} we can then construct the denominator $Q_n(z; \rho_{2n})$ in a coprime factorization of the transfer function $W_n(z; \rho_{2n})$, again using (12b) with $k = n-1$:

$$Q_n(z) = (z - \rho_{2n})Q_{n-1}(z) - \rho_{2n-1}Q_{n-2}(z). \tag{12b)'$$

In (12b), ρ_{2n-1}, Q_{n-1} and Q_{n-2} are fixed and Q_n depends (as a monic polynomial of degree n) only on ρ_{2n}. Consider the curve $C(\gamma)$, which is defined as the image of the imaginary axis under the real rational function

$$f(z) = Q_{n-1}(z)/[zQ_{n-1}(z) - \rho_{2n-1}Q_{n-2}(z)].$$

Note that f is strictly proper of degree n, since $\rho_{2n-1} \neq 0$ and Q_{n-1} and Q_{n-2} are relatively prime (Theorem 3).

Theorem 5. *Suppose $f(z)$ has no pole on the imaginary axis and that $1/\rho_{2n}$ does not lie on the contour $C(\gamma)$. If ζ denotes the winding number, computed in the counterclockwise orientation of $C(\gamma)$ about $1/\rho_{2n}$, and ν denotes the number of zeroes of Q_{n-1} in the left-half plane, then $Q_n(z; \rho_{2n})$ has $\zeta + \nu$ zeroes in the left-half plane. In particular, the minimal partial realization of $\gamma = (\gamma_1, \ldots, \gamma_{2n-1})$ corresponding to the parameter ρ_{2n} is stable if, and only if,*

$$\zeta = n - \#\{\text{left-half plane zeroes of } Q_{n-1}\}.$$

Proof. The open loop transfer function $f(z)$, under scalar gain output feedback k, transforms into the closed loop function:

$$f^k(z) = \frac{Q_{n-1}(z)}{(z+k)Q_{n-1}(z) - \rho_{2n-1}Q_{n-2}(z)}.$$

Setting $k = -\rho_{2n}$, the classical Nyquist criterion asserts that $\zeta + \nu$ is the number of zeroes of

$$(z - \rho_{2n})Q_{n-1}(z) - \rho_{2n-1}Q_{n-2}(z).$$

Comparing with (12b) yields the conclusion. □

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