Robust Control with Classical Methods – QFT

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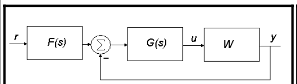
- Review of the classical Bode-Nichols control problem
- QFT in the basic Single Input Single Output (SISO) case
- Uncertainty and Fundamental Design Limitations
- QFT for non-minimum phase and computer controlled systems
- QFT for cascaded systems, and for a class of non-linear plants
- QFT for Multi-Input Multi-Output (MIMO) plants
- A comparison between QFT and other robust and adaptive control

QFT for a class of non-linear plants

- **The principle**
- **To obtain the LTIE set**
  - The Barrel analogy
  - Examples
- **LTI design**
- **Convergence for the non-linear system**
  - use of Brouwer fixed point theorem
- **Exercise**
- **Non-zero initial conditions, ...**
- **Cancellation in non-linear QFT**
- **Design of Λ**
- **Example**

The principle



- $W \in \mathcal{W}$ is a non-linear plant
- **Objective**: For a specific input $r(t)$ it is required that the output $y(t)$ be a member of an acceptable set $\mathcal{Y}$ for all $W \in \mathcal{W}$.

- **Procedure**: Replace the non-linear plant set $\mathcal{W}$ by a linear time invariant equivalent LTIE set $\mathcal{L}$.
- **Assumptions** in the basic case treated here:
  - Zero initial conditions
  - No disturbance inputs
- **Reference**: Horowitz, ch 11

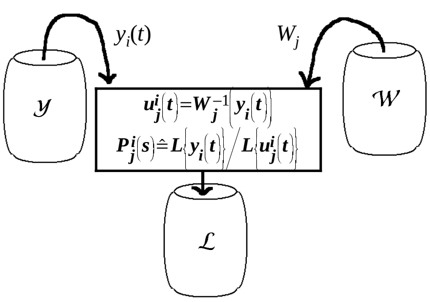
To obtain the LTIE set

- Choose one $y_i(t) \in \mathcal{Y}$, and one plant $W_j \in \mathcal{W}$, and solve for the plant input,

$$u_j^i(t) = W_j^{-1}\{y_i(t)\}$$

- Define $\boxed{P_j^i(s) \triangleq L\{y_i(t)\} / L\{u_j^i(t)\}}$ as the LTIE of W_j w.r.t. $y_i(t)$.
- $L\{y_i(t)\}$ must exist
- For given $y_i(t)$, $u_j^i(t) = W_j^{-1}\{y_i(t)\}$ must be **unique** and Laplace transformable.

The barrel analogy



Examples

- $\mathcal{W}$: $y=4u^2$
- $\mathcal{Y}$: $\{a, t>0; \text{else } 0\}$, $a \in [1,9]$
- $y=a \Rightarrow u=\sqrt{a}/2$
- $P_{eq}=y/u= 2\sqrt{a} \in [2,6]$
- Note: even w/o uncertainty in $\mathcal{W}$, P_{eq} becomes uncertain!

- $\mathcal{W}$: $y=ku^2$, $k \in [1,4]$
- $\mathcal{Y}$: $\{a, t>0; \text{else } 0\}$, $a \in [1,9]$
- $y=a \Rightarrow u=\sqrt{a}/\sqrt{k}$
- $P_{eq}=y/u=\sqrt{(ka)} \in [1,6]$

Examples, cont'd

- $\mathcal{W}$: $\dot{y}(t)+Hy^2(t)=Ku(t),\ H\in[H_1,H_2],\ K\in[K_1,K_2],\ \dot{y}(0)=y(0)=0$
- $\mathcal{Y}$: $\{Ate^{-\alpha t},\ A\in[1,2],\ \alpha\in[1.5,3]\}$

$$sY(s)+HL\{y^2(t)\}=KU(s) \qquad \Rightarrow P_{eq}(s)=\frac{Y(s)}{U(s)}=\frac{K}{s+\left(HL\{y^2(t)\}/Y(s)\right)}$$

$$Y(s)=A/(s+\alpha)^2$$

$$y^2(t)=A^2t^2e^{-2\alpha t} \qquad L\{y^2(t)\}=2A^2/(s+2\alpha)^3$$

$$P_{eq}(s)=\frac{K}{s+\frac{2HA^2}{(s+2\alpha)^3}\frac{(s+\alpha)^2}{A}}=\frac{K(s+2\alpha)^3}{s(s+2\alpha)^3+2HA(s+\alpha)^2}$$

LTI design

- With $P_{eq}(s)$ defining a set of LTI transfer function, design the feedback compensator $G(s)$ and prefilter $F(s)$ such that the closed loop around $P^i_j(s)$ satisfies the specifications defined by $\mathcal{Y}$ and corresponding $r(t)$.
- Will the closed loop around $W\in\mathcal{W}$ also satisfy the specifications?

Convergence for the non-linear system

- Define a map Φ: $Y(j\omega)\to\mathbb{C}$, with $F(s)$, $G(s)$ continuous,

$$\Phi(Y)=\frac{P^i_j(s)G(s)F(s)}{1+P^i_j(s)G(s)}$$

- **Suppose** that there exists a fixed point, i.e. $Y^*=\Phi(Y^*)$
- Recall that $P^i_j(s)=Y_i(s)/U^i_j(s) \Rightarrow$

$$Y^*(s)=\frac{Y^*(s)G(s)F(s)}{U^i_j(s)+Y^*(s)G(s)},\ s=j\omega$$

- then Y^* is the output of the closed loop system around W.

Non-linear convergence, cont'd

- **If** there is a pair $F(s)$, $G(s)$ such that Φ has a fixed point for all $P^i_j(s)$, then the output of $W\in\mathcal{W}$ belongs to the acceptable set $\mathcal{Y}$.
- **Conditions for the existence of fix points (Special case of Shauder's f.p.t. called Brouwer's f.p.t)**:
  1) $L\{\mathcal{Y}\}$ convex, compact; $Y(j\omega)$ is continuous in ω
  1) Φ: $L\{\mathcal{Y}\}\to L\{\mathcal{Y}\}$
  2) Φ is continuous
  3) $1+P^i_j(s)G(s)$ are analytic in RHP.

When do $F(s)$ and $G(s)$ exist such that a fix point is guaranteed?

1. $P^i_j(s)$ is minimum phase, and $u^i_j(t)$ is uniformly bounded
2. A uniform bound exists for the unstable poles of $P^i_j(s)$
3. As $s\to\infty$, $P^i_j(s)\to k^i_j/s^{e^i_j}$ where $\Delta e^i_j\le 1$ (may be relaxed)

Exercise: van der Pol plant

- $\mathcal{W}$: $\ddot{y}+A\dot{y}(By^2-1)+Ey=Ku,\quad \ddot{y}(0)=\dot{y}(0)=y(0)=0$
  $A\in[1,3],\ B\in[1,4],\ E\in[-2,1],\ K\in[31,124]$
- $\mathcal{Y}$: $Y(s)=T(s)R(s)$

$$T(s)=\frac{480}{z}\frac{(s+z)}{(s+2)(s+3)(s+4)(s+20)},\ z\in[1,100]$$

$$R(s)=Q/s,\ z\in[0,2]$$

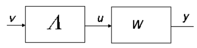
- Find $P_{eq}(s)$, and design $G(s)$ and $F(s)$!

Non-zero initial conditions, disturbance inputs, ...

- See Horowitz, ch. 11

Cancellation in non-linear QFT

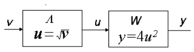
- **Problem**: In the ”Barrel Analogy” method, the LTIE plant set, P_{eq}, is uncertain even if W is certain.
- **Solution**: ”Cancellation” at the nominal plant.



Choose a nominal plant W_0, and find a network Λ such that $W_0(\Lambda(v)) = L^{-1}\{H(s)V(s)\}$, where $H(s)$ is fixed LTI.

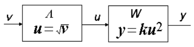
Cancellation, example

- $\mathcal{W}$: $y=4u^2$
- $\mathcal{Y}$: $\{a, t>0; \text{ else } 0\}$, $a\in[1,9]$
- $y=a \Rightarrow u=\sqrt{a}/2$
- $P_{eq}=y/u= 2\sqrt{a} \in[2,6]$



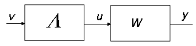
- $P_{eq}{}' = y/v=4$

- $\mathcal{W}$: $y=ku^2$, $k \in[1,4]$
- $\mathcal{Y}$: $\{a, t>0; \text{ else } 0\}$, $a\in[1,9]$
- $y=a \Rightarrow u=\sqrt{a}/\sqrt{k}$
- $P_{eq}=y/u=\sqrt{(ka)}\in[1,6]$

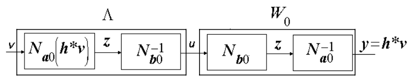


- $P_{eq}{}'=y/v= k \in[1,4]$

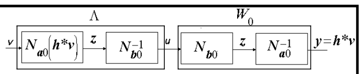
Design of Λ



- Desired: $y(t)= h(t)*v(t)$ or $Y(s)= H(s)U(s)$
- The non-linear plant is described by
  W: $N_a y=N_b u=z$, or $y=N_a^{-1}N_b u$, or $u=N_b^{-1}N_a y$



- Note: with zero i.c., $L\{\dot{y}\}=sH(s)V(s) \Rightarrow \dot{y}=\dot{h}*v$, ...



Example

$\mathcal{W}$: $N_a y=\dot{y}+\Psi y^2\text{sign}(y)=ku=N_b u$, $\Psi\in[1,6]$, $k\in[1,2]$

$\mathcal{Y}$: $Y(s)= T(s)R(s)$

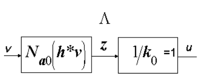
$R(s)=M/s$, $M\in[-10,10]$

$T(s)=\omega_n^2/(s^2+2\zeta\omega_n s+\omega_n^2)$, $\omega_n\in[0.7,4]$, $\zeta\in[0.7,1.2]$

- **Design of Λ**

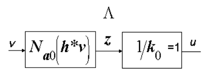
Choose nominal $\psi_0=6$, $k_0=1$

$N_{b0}u=k_0u=z \Rightarrow u=(1/k_0)z=N_{b0}^{-1}z$

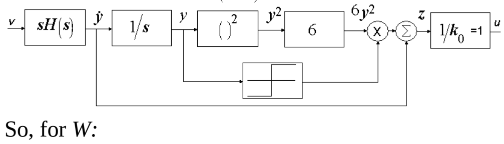


$N_{a0}y=N_{a0}(h*v)=(\dot{h}*v)+6(h*v)^2\text{sign}(h*v)=z$

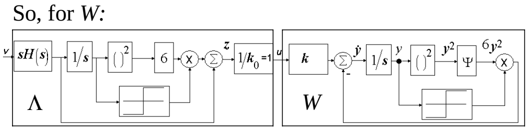
Example, cont’d



... for W_0 : $N_{a0}y=N_{a0}(h*v)=(\dot{h}*v)+6(h*v)^2\text{sign}(h*v)=z$ giving Λ:



- So, for W:



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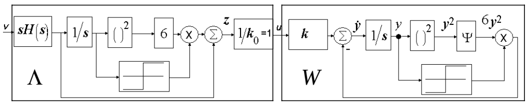
Example, cont'd



- The equation for the plant with cancellation, $y=W\Lambda v$:
  $y=\dot{y}+\Psi y^2\text{sign}(y)=(k/k_0)\left(\dot{h}*v+6(h*v)^2\text{sign}(h*v)\right)$, $\Psi\in[1,6]$, $k\in[1,2]$
- Find LTIE $Y(s)/V(s)$ as above.
- Alternatively, find LTIE $Y(s)/X(s)$ with $X(s)=H(s)V(s)$
- Note that $H(s)$ is known, and arbitrary.

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Example, cont'd

- Templates with k=1

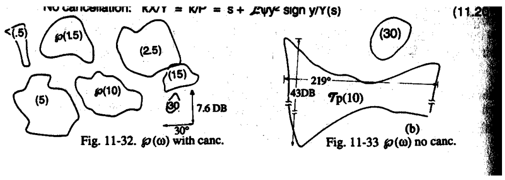

Fig. 11-32. ℘(ω) with canc. Fig. 11-33 ℘(ω) no canc.

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Example, cont'd

$\mathscr{Y}$: $Y(s)=T(s)R(s)$, $R(s)=M/s$, $M\in[-10,10]$

$T(s)=\omega_n^2/(s^2+2\zeta\omega_n s+\omega_n^2)$, $\omega_n\in[0.7,4]$, $\zeta\in[0.7,1.2]$

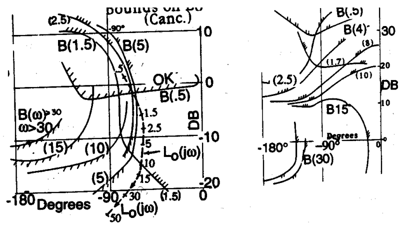

Fig. 11-34 a, b. L_o Bounds with and without canc.