SF2524 Matrix Computations for Large-scale Systems
Exam

Aids: None   Time: Four hours

Grades: E: 17 points, D: 19 points, C: 22 points, B: 26 points, A: 29 points (out of the possible 33 points, including bonus points from homeworks).

Problem 1 (0p) How many bonus points did you accumulate in HT2018? (max=3)

Problem 2 (5p)
In the program to the right (which is a partial implementation of phase 2 of the QR-algorithmm) we want the result to be an upper triangular matrix.

(a) What should "??" be?

(b) Even if the "??" are filled out correctly, there is a bug / error. What is it?

(c) How can the QR-factorization of `H` be obtained from the code?

```
n=100; A=randn(n);
H=hess(A); % Create a hessenberg matrix
for k=1:n
   r=??
   c=??
   s=??
   G=eye(n); G([k:k+1],[k:k+1])=[c -s; s c];
   H=G'*H
end
```

Solution:

(a)
```
r=sqrt(H(k,k)^2+H(k+1,k)^2)
c=H(k,k)/r
s=H(k+1,k)/r;
```

(b) The loop should run to $n-1$

(c) The QR-factorization can be obtained: R: is the `H` after the algorithm
Q: is the product of the givens rotators

$$Q = G_1 \cdots G_{n-1}$$

where G_k is the `G`-matrix created in loop k. (It can also be more efficiently computed by only computing actions of the givens rotators.)

Problem 3 (5p)

(a) What is the minimization definition of the GMRES-iterates?

(b) Describe in MATLAB (or Julia-code) the GMRES method.

(c) Suggest one way to improve the method when the matrix is symmetric.

Solution:

(a) Minimization definition of GMRES

$$\|Ax_m - b\|_2 = \min_{x \in \mathcal{K}_m} \|Ax - b\|_2$$

where x_m is the iterate at step m.

(b) See lecture notes.

(c) The method can be improved if the matrix is symmetric since then the orthogonalization in the Arnoldi method can be done with a short-term recurrence. In other words, Arnoldi is replaced by Lanczos.

Problem 4 (5p) The φ-function is defined as $\varphi(z) = \frac{e^z - 1}{z}$.

(a) How can the matrix φ-function be used to solve the inhomogeneous ODE $y'(t) = Ay(t) + b$?

(b) The matrix φ-function can be computed by using the matrix exponential as follows

$$\varphi(A) = \begin{bmatrix} I_n & ?? \end{bmatrix} \exp \begin{bmatrix} A & ?? \\ 0_n & 0_n \end{bmatrix} \begin{bmatrix} ?? \\ ?? \end{bmatrix}$$

where $I_n \in \mathbb{R}^{n \times n}$ is the identity matrix and $0_n \in \mathbb{R}^{n \times n}$ is a zero matrix. Prove the formula and show what the ?? should be.

Solution:

(a) This gives directly the solution to the linear inhomogeneous ODE:

$$y(t) = y(0) + t\varphi(tA)(Ay(0) + b).$$

(b) We use the technique we used in the derivation of Schur-Parlett. Let

$$E = \begin{bmatrix} E_1 & E_2 \\ 0 & E_4 \end{bmatrix} := \exp\left(\begin{bmatrix} A & X \\ 0 & 0 \end{bmatrix}\right)$$

where we want to determine X. From matrix functions on a block triangular matrix we know that

$$E_1 = \exp(A), \quad E_4 = \exp(0) = I.$$

Since the $f(B)$ always commutes with B we have

$$\begin{bmatrix} E_1 & E_2 \\ 0 & E_4 \end{bmatrix} \begin{bmatrix} A & X \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A & X \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_1 & E_2 \\ 0 & E_4 \end{bmatrix}.$$

The (2,2)-block equation becomes after some manipulations

$$\begin{aligned} E_1 X &= A E_2 + X E_4 \\ \exp(A) X &= A E_2 + X \\ E_2 &= A^{-1}(\exp(A) - I) X \end{aligned}$$

We see that $E_2 = \varphi(A)$ if $X = I$. Hence

$$\varphi(A) = \begin{bmatrix} I_n & 0 \end{bmatrix} \exp \begin{bmatrix} A & I_n \\ 0_n & 0_n \end{bmatrix} \begin{bmatrix} 0 \\ I_n \end{bmatrix}$$

Problem 5 (5p) Let $A \in \mathbb{R}^{n \times n}$ be triangular diagonalizable matrix with diagonal elements $a_{i,i} = \sqrt{i}$. Let $f(x) = 1/(10 + x)$.

(a) What is the Jordan form definition of $f(A)$?

(b) For what values of n is the matrix function defined/finite?

Solution:

(a) The Jordan definition is

$$f(A) = V f(J) V^{-1}$$

where $f(J) = \operatorname{diag}(f(J_1), \ldots, f(J_k))$ and

$$f(J_k) = \begin{bmatrix} f(\lambda) & \cdots & f^{(p-1)}(\lambda)/(p-1)! \\ 0 & \ddots & \vdots \\ 0 & 0 & f(\lambda) \end{bmatrix}$$

(b) The jordan form is defined for all n, since $f(x)$ is analytic except for a pole in $x = -10$, which is never an eigenvalue of this matrix (they are $\sqrt{1}, \sqrt{2}, \ldots$.)

Problem 6 (5p) We are applying CG to a matrix with one eigenvalue equal to $\lambda_1 = 10$, and all the other eigenvalues $\lambda_2, \ldots, \lambda_n \in [1 - \varepsilon, 1 + \varepsilon]$.

(a) Derive a bound on the error (using any theorem in the course) for $\varepsilon = 1/2$ which does not depend on n .

(b) Derive a bound from the min-max error characterization

$$\min_{p \in P_m^0} \max_{i=1,..,n} |p(\lambda_i)|.$$

which shows that when $m = 2$ the error can be bounded essentially proportional to ε, when ε is small.

Problem 7 (5p) We apply the power method to the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3+\varepsilon \end{bmatrix}$$

with starting vector $[1, 1]^T$.

(a) To what eigenvalue does the method converge, and what is the convergence factor when $\varepsilon > 0$. You may use any theorem from the course.

(b) Prove that the method converges also when $\varepsilon = 0$. Hint: Use the Jordan definition of matrix functions.

Solution:

(a) The eigenvalues of the matrix are 3 and $3 + \varepsilon$. The power method in general converges to the largest eigenvalue, which is $3 + \varepsilon$. The convergence rate is given by the quotient of the eigenvalues in modulus. The eigenvector error is

$$\left(\frac{3}{3+\varepsilon}\right)^k$$

and the eigenvalue error is $(3^2/(3 + \varepsilon)^2)^k$.

(b) The power method applied to the case $\varepsilon = 0$ can be expressed as

$$v_k = \frac{w_k}{\|w_k\|}$$

where

$$w_k = A^k v_0.$$

From the matrix function definition of A^k, we have with $f(z) = z^k$ that

$$f(A) = \begin{bmatrix} f(3) & f'(3) \\ 0 & f(3) \end{bmatrix} = \begin{bmatrix} 3^k & k3^{k-1} \\ 0 & 3^k \end{bmatrix}$$

We factorize the dominant term (as in the standard derivation of the power method). This implies that

$$w_k = \begin{bmatrix} 3^k & k3^{k-1} \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3^k + k3^{k-1} \\ 3^k \end{bmatrix} = (3^k + k3^{k-1}) \begin{bmatrix} 1 \\ \alpha_k \end{bmatrix}$$

where $\alpha_k = 3^k/(3^k + k3^{k-1})$. Hence,

$$v_k = \frac{1}{\|w_k\|} w_k = \frac{1}{\|w_k/(3^k + k3^{k-1})\|} w_k/(3^k + k3^{k-1}) = \frac{1}{\| \left[\begin{smallmatrix} 1 \\ \alpha_k \end{smallmatrix}\right] \|} \begin{bmatrix} 1 \\ \alpha_k \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

since

$$\alpha_k = \frac{1}{1 + k/3} \to 0 \text{ as } k \to \infty.$$

The vector $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ is an eigenvector corresponding to the eigenvalue $\lambda = 3$.