SF2524 Matrix Computations for Large-scale Systems
Exam

Aids: None   Time: Four hours

Grades: E: 17 points, D: 19 points, C: 22 points, B: 26 points, A: 29 points (out of the possible 33 points, including bonus points from homeworks).

Problem 1 (0p) How many bonus points did you accumulate in HT2018? (max=3)

Problem 2 (5p)
In the program to the right (which is a partial implementation of phase 2 of the QR-algorithmm) we want the result to be an upper triangular matrix.

(a) What should "??" be?

(b) Even if the "??" are filled out correctly, there is a bug / error. What is it?

(c) How can the QR-factorization of `H` be obtained from the code?

```
n=100; A=randn(n);
H=hess(A); % Create a hessenberg matrix
for k=1:n
   r=??
   c=??
   s=??
   G=eye(n); G([k:k+1],[k:k+1])=[c -s; s c];
   H=G'*H
end
```

Problem 3 (5p)

(a) What is the minimization definition of the GMRES-iterates?

(b) Describe in MATLAB (or Julia-code) the GMRES method.

(c) Suggest one way to improve the method when the matrix is symmetric.

Problem 4 (5p) The φ-function is defined as $\varphi(z) = \frac{e^z-1}{z}$.

(a) How can the matrix φ-function be used to solve the inhomogeneous ODE $y'(t) = Ay(t) + b$?

(b) The matrix φ-function can be computed by using the matrix exponential as follows

$$\varphi(A) = \begin{bmatrix} I_n & ?? \end{bmatrix} \exp \begin{bmatrix} A & ?? \\ 0_n & 0_n \end{bmatrix} \begin{bmatrix} ?? \\ ?? \end{bmatrix}$$

where $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix and $0_n \in \mathbb{R}^{n\times n}$ is a zero matrix. Prove the formula and show what the ?? should be.

Problem 5 (5p) Let $A \in \mathbb{R}^{n\times n}$ be triangular diagonalizable matrix with diagonal elements $a_{i,i} = \sqrt{i}$. Let $f(x) = 1/(10 + x)$.

(a) What is the Jordan form definition of $f(A)$?

(b) For what values of n is the matrix function defined/finite?

Problem 6 (5p) We are applying CG to a matrix with one eigenvalue equal to $\lambda_1 = 10$, and all the other eigenvalues $\lambda_2, \dots, \lambda_n \in [1 - \varepsilon, 1 + \varepsilon]$.

(a) Derive a bound on the error (using any theorem in the course) for $\varepsilon = 1/2$ which does not depend on n .

(b) Derive a bound from the min-max error characterization

$$\min_{p\in P_m^0} \max_{i=1,..,n} |p(\lambda_i)|.$$

which shows that when $m = 2$ the error can be bounded essentially proportional to ε, when ε is small.

Problem 7 (5p) We apply the power method to the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 + \varepsilon \end{bmatrix}$$

with starting vector $[1, 1]^T$.

(a) To what eigenvalue does the method converge, and what is the convergence factor when $\varepsilon > 0$. You may use any theorem from the course.

(b) Prove that the method converges also when $\varepsilon = 0$. Hint: Use the Jordan definition of matrix functions.