SF2524 Matrix Computations for Large-scale Systems
Exam

Aids: None   Time: Four hours

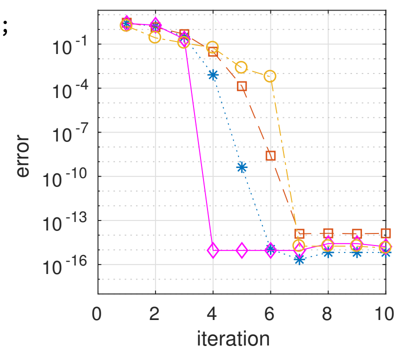
Grades: E: 17 points, D: 19 points, C: 22 points, B: 26 points, A: 29 points (out of the possible 33 points, including bonus points from homeworks).

Problem 1 (5p)

(a) What should ?? be to obtain the dotted curve with stars. Relate to theory for Rayleigh quotient. (Note: Answer without relation to theory will not give full points).

(b) Which one of curves corresponds to Arnoldi's method for eigenvalue problems applied to A with `b=eye(n,1)` and ?? selected as -5. Relate to theory for Arnoldi's method.

```
>> A=[1 3 3 ??; 3 -1 3 2;
  3 3 4 -1; 4 2 -1 -1];
>> x=ones(4,1); s=0;
>> for k=1:10
>>   x=(A-s*eye(4))\x;
>>   x=x/norm(x);
>>   s=x'*A*x;
>> end
```



Problem 2 (6p)

(a) State the minimization definition of the CG-iterates.

(b) How are the iterates of CGN (sometimes called CGNE) defined?

(c) Prove that the CGN iterates are minimizers of a residual with respect to a norm over a space X. Which norm and what is X?

Problem 3 (5p) In this question you shall apply a result for the movement (perturbation) of eigenvalues known as the Bauer-Fike theorem:

> The eigenvalues of $A = B + C$ are contained in discs centered at the eigenvalues of B with radius $K\|C\|$.

where K is the eigenvalue condition number $K = \|V\|\|V^{-1}\|$. We apply GMRES to the matrix $A = \alpha I + C$, where A is a normal matrix such that $\|V\|\|V^{-1}\| = 1$. Provide a convergence factor bound in terms of α and $\|C\|$. You may invoke any theorem/lemma we have used in the course.

Problem 4 (5p) We define the scalar product $< u, v >= u^T L^T L v$ for some non-singular matrix L. We say that a matrix $Q = [q_1, \ldots, q_m]$ is orthogonal with respect to this scalar product if

$$< q_i, q_j >= \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

which is equivalent to the equation $Q^T L^T L Q = I$.

(a) Suppose $Q = [q_1, \ldots, q_m]$ is orthogonal with respect to this scalar product. Construct a method of the type

```
>>     h=??
>>     z=??
>>     beta=??
>>     q_new=z/beta;
```

where all operations are done using matrices. The method should, given a vector $b \notin \text{span}(q_1, \ldots, q_m)$, construct $q_{m+1} :=$`q_new` such that it satisfies $< q_i, q_{m+1} >= 0$ for $i = 1, \ldots, m$ and $< q_{m+1}, q_{m+1} >= 1$ and $b = h_1 q_1 + \ldots + h_m q_m + \beta q_{m+1}$.

(b) We now apply Arnoldi's method with the orthogonalization procedure stated in (a). Let Q_m and $\underline{H}_m$ be the matrices generated. Suppose the matrix A is not symmetric but satisfies instead $AL^T L = L^T L A^T$ which means $< u, Av >=< Au, v >$ for any u and v. What property/structure of H_m does this imply?

Problem 5 (4p) Let

$$P = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}, \qquad f(P) = F = \begin{bmatrix} F_A & F_B \\ 0 & F_C \end{bmatrix}$$

where $C = \alpha I$ and $A, B, C, F_A, F_B, F_C \in \mathbb{C}^{n \times n}$. Derive a formula for F_C only involving $A, B, \alpha, f(A)$, and $f(B)$.

Problem 6 (5p)

(a) Prove shift invariance of Arnoldi factorization by showing a relation of the form

$$(A - \mu I) Q_m = Q_{m+1}???$$

(b) In a particular application we discretize a parameter dependent PDE which leads to a parameter dependent linear system of equations $g(\mu) = (A - \mu I)^{-1} b$ where $A \in \mathbb{R}^{n \times n}$ is large and sparse. We want to evaluate $g(\mu)$ for $\mu = \mu_1, \ldots, \mu_p$ for a large p-value. Derive and explain (for instance in the form of a program) a method based on GMRES, which only requires the computation of $N = 100$ matrix-vector products with matrix A, to compute *all* vectors $g(\mu_1), \ldots, g(\mu_p)$. (That is, the number of matrix-vector products with A is independent of p.) You may assume that GMRES converges in N steps.