ROYAL INSTITUTE OF TECHNOLOGY

# Fourier integrals and the sampling theorem

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## Fourier Integrals

Read: Strang, Section 4.3.
Fourier series are convenient to describe *periodic* functions
(or functions with support on a finite interval).
What to do if the function is not periodic?

- Consider $f : \mathbb{R} \to \mathbb{C}$. The *Fourier transform* $\hat{f} = \mathcal{F}(f)$ is given by

$$\boxed{\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx, \quad k \in \mathbb{R}.}$$

  Here, $f$ should be in $L^1(\mathbb{R})$.
- Inverse transformation:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}\,dk$$

  Note: Often, you will se a more symmetric version by using a different scaling.
- Theorem of Plancherel: $f \in L^2(\mathbb{R}) \Leftrightarrow \hat{f} \in L^2(\mathbb{R})$ and

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(k)|^2 dk.$$

## Fourier Integrals: The Key Rules

$$\widehat{df/dx} = ik\hat{f}(k) \qquad \widehat{\int_{-\infty}^{x} f(\xi)d\xi} = \hat{f}(k)/(ik)$$

$$\widehat{f(x-d)} = e^{-ikd}\hat{f}(k) \qquad \widehat{e^{icx}f} = \hat{f}(k-c)$$

**Examples of Fourier transforms:**

$$f(x) = sin(\omega x) \qquad \hat{f}(k) = -i\pi(\delta(k-\omega) - \delta(k+\omega))$$

$$f(x) = cos(\omega x) \qquad \hat{f}(k) = \pi(\delta(k-\omega) + \delta(k+\omega))$$

$$f(x) = \delta(x) \qquad \hat{f}(k) = 1 \text{ for all } k \in \mathbb{R}.$$

$$f(x) = \begin{cases} 1, & \text{if } -L \le x \le L, \\ 0, & \text{if } |x| > L. \end{cases} \qquad \hat{f}(k) = 2\frac{\sin kL}{k} = 2L\text{sinc}(kL),$$

where $\text{sinc}(t) = \sin(t)/t$ is the *Sinus cardinalis* function.

## Sampling of functions - The Nyquist sampling rate

How many samples do we need to identify $\sin \omega t$ or $\cos \omega t$?
Assume equidistant sampling with *step size* $T$.

| *Definition:* Nyquist sampling rate: $T = \pi/\omega$. |
|---|

The Nyquist sampling rate, is exactly 2 equidistant samples over a full period of the function. (period is $2\pi/\omega$).

For a given sampling rate $T$, frequencies higher than the *Nyquist frequency* $\omega_N = \pi/T$ cannot be detected. A higher frequency harmonic is mapped to a lower frequency one. This effect is called *aliasing*.

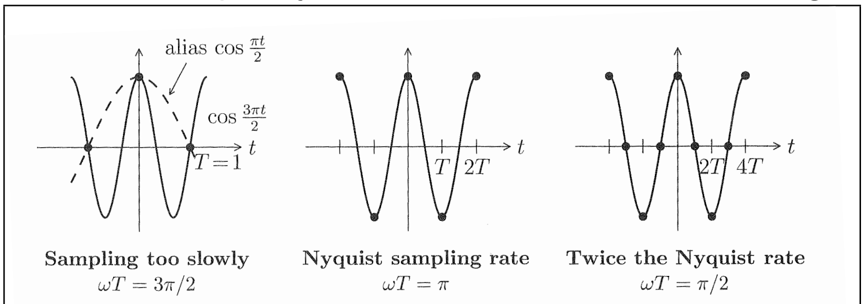


Figure B1: For $\omega T < \pi$, the discrete samples are enough to determine the signal.

Figure from Strang.

## Band-limited functions

- When the continuous signal is a combination of many frequencies $\omega$, the largest frequency $\omega_{max}$ sets the Nyquist sampling rate at $T = \pi/\omega_{max}$.
- No aliasing at any faster rate. (Do not set sampling rate equal to the Nyquist sampling rate. Consider $\sin \omega T$ to see why.)
- A function $f(x)$ is called band-limited if there is a finite $W$ such that its Fourier integral transform $\hat{f}(k) = 0$ for all $|k| \geq W$.

The Shannon-Nyquist sampling theorem states that such a function $f(x)$ can be recovered from the discrete samples with sampling frequency $T = \pi/W$.

(Note that relating to above, $W = \omega_{max} + \varepsilon$, $\varepsilon > 0$. )

## The Sampling Theorem

**Theorem:** (Shannon-Nyquist) Assume that $f$ is band-limited by $W$, i.e., $\hat{f}(k) = 0$ for all $|k| \geq W$. Let $T = \pi/W$ be the Nyquist rate. Then it holds

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} f(nT)\text{sinc}(\pi(x/T - n))}$$

where $\text{sinc}(t) = \sin(t)/t$ is the *Sinus cardinalis* function.
Note: The sinc function is band-limited:

$$\widehat{\text{sinc}}(k) = \begin{cases} 1, & \text{if } -\pi \leq k \leq \pi, \\ 0, & \text{elsewhere.} \end{cases}$$

Also note: This interpolation procedure is not used in practice. Slow decay, summing an infinite number of terms. Approximations to the sinc function are used, introducing interpolation errors.

# The Sampling Theorem: Proof

Assume for simplicity $W = \pi$.
By the inverse Fourier transform,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}dk = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}(k)e^{ikx}dk.$$

Define

$$\tilde{f}(k) = \begin{cases} \hat{f}(k), & \text{if } -\pi < k < \pi, \\ \text{periodic continuation}, & \text{if } |k| \geq \pi. \end{cases}$$

$\tilde{f}$ can be represented as a Fourier series in $k$-space,

$$\tilde{f}(k) = \sum_{n=-\infty}^{\infty} \hat{\hat{f}}_n e^{ink},$$

where

$$\hat{\hat{f}}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{f}(k)e^{-ink}dk = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}(k)e^{-ink}dk = f(-n).$$

Hence, for $-\pi < k < \pi$,

$$\hat{f}(k) = \tilde{f}(k) = \sum_{n=-\infty}^{\infty} f(-n)e^{ink}.$$

# The Sampling Theorem: Proof, contd.

Therefore,

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}(k)e^{ikx}dk \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sum_{n=-\infty}^{\infty} f(-n)e^{ink}\right) e^{ikx}dk \\ &= \sum_{n=-\infty}^{\infty} f(-n)\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik(x+n)}dk \\ &= \sum_{n=-\infty}^{\infty} f(-n)\frac{\sin\pi(x+n)}{\pi(x+n)} \\ &= \sum_{n=-\infty}^{\infty} f(n)\frac{\sin\pi(x-n)}{\pi(x-n)} \end{aligned}$$

If $W$ different from $\pi$, rescale $x$ by $\pi/W$ and $k$ by $W/\pi$ to complete the proof.