KTH Matematik

EXAMINATION IN SF2980 RISK MANAGEMENT, 2009-12-18, 14:00–19:00.

Examiner: Henrik Hult, tel. 790 6911, e-mail: hult@kth.se

Allowed technical aids: calculator.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow.

GOOD LUCK!

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Problem 1

Suppose you have observations of the independent random variables $X_1, \dots, X_{1000}$. All the X_i's have the same distribution with unknown continuous distribution function F. Explain in detail how to construct an exact (two-sided) confidence interval for the 0.99-quantile of F and how to compute the exact confidence level. (10 p)

Problem 2

Suppose you have observations of the independent random variables $X_1, \dots, X_{100}$. All the X_i's have the same distribution with unknown continuous distribution function F. Consider estimating the quantity

$$d = \frac{1}{0.02} \int_{0.98}^{1} F^{-1}(u) du,$$

based on $X_1, \dots, X_n$.

(a) Explain the empirical method for computing an estimate of d and be sure to mention the underlying assumptions. Explain also how to implement the method. Discuss its advantages and disadvantages. (5 p)

(b) Explain the Peaks-Over-Threshold method for computing an estimate of d and be sure to mention the underlying assumptions. Explain also how to implement the method and how to check the validity of the underlying assumptions. Discuss its advantages and disadvantages. (5 p)

Problem 3

Suppose (U, V) has joint distribution function $C(u, v)$ which is a copula. Let $(X, Y) = (\Phi^{-1}(1-U), \Phi^{-1}(V))$, where Φ is the distribution function of the standard normal distribution. Determine the lower tail dependence coefficient $\lambda_L(X, Y)$

(a) when $C(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}, \quad \theta > 0,$ (5 p)

(b) when $C(u, v) = C_R^{Ga}(u, v)$ is a Gaussian copula with (5 p)

$$R = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}.$$

Problem 4

The historical daily log-returns of two assets, called asset A and asset B, are plotted in Figure 1 and a quantile-quantile plot of the emprical quantiles of the daily log-returns is given in Figure 2. Suppose today's asset prices are $S_A = 100$ for asset A and $S_B = 300$ for asset B. Consider the quantile at level 0.95 of the linearized portfolio loss over a one-day time horizon. Which of the following three portfolios has the highest and which has the lowest quantile?

1. Buy three share of asset A and one share of asset B.
2. Buy six shares of asset A (no shares of asset B).
3. Buy two shares of asset B (no shares of asset A).

Any assumptions must be properly stated. (10 p)

Problem 5

Consider a latent variable model in portfolio credit risk. Suppose there are n obligors in the portfolio. Each obligor is associated with a latent variable Y_i, $i = 1, \ldots, n$ with continuous distribution function F_{Y_i}. Suppose that there are independent (not necessarily identically distributed) random variables Z and $Z_1, \ldots, Z_n$ with continuous distribution functions F_Z and F_{Z_i} such that the latent variables can be represented as $Y_i = Z + Z_i$ for each $i = 1, \ldots, n$. Let $p_1, \ldots, p_n$ be the individual default probabilities. The default indicators X_i are defined to be 1 if $Y_i \leq d_i$ and 0 otherwise. The level d_i is determined by the default probability.

(a) Show that the default indicators X_i are conditionally independent given Z. (5 p)

(b) Find functions $f_i(Z)$, $i = 1, \ldots, n$ such that the default indicators $(X_1, \ldots, X_n)$ can be represented as a Bernoulli mixture model with $P(X_i = 1 \mid Z) = f_i(Z)$. (5 p)

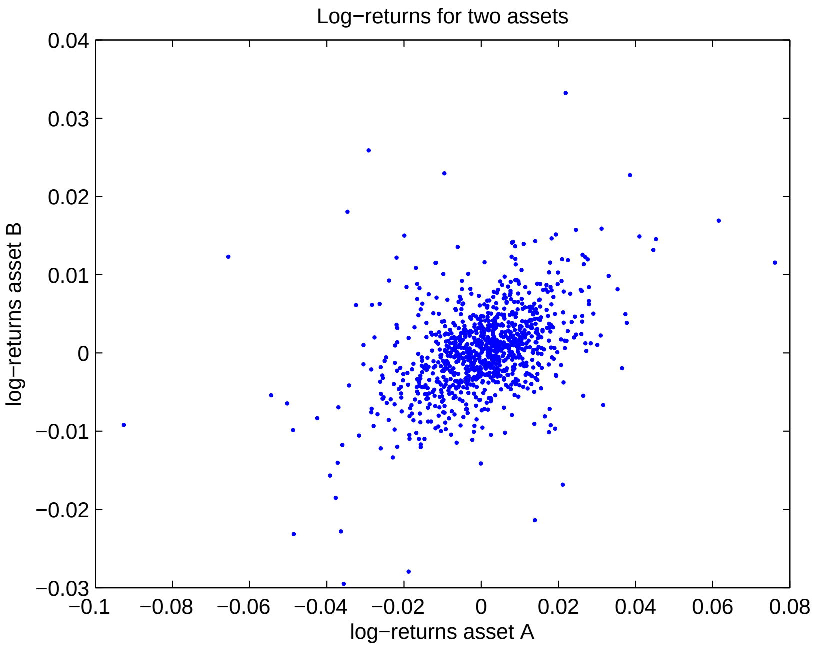

Figure 1: Scatter plot of log-returns

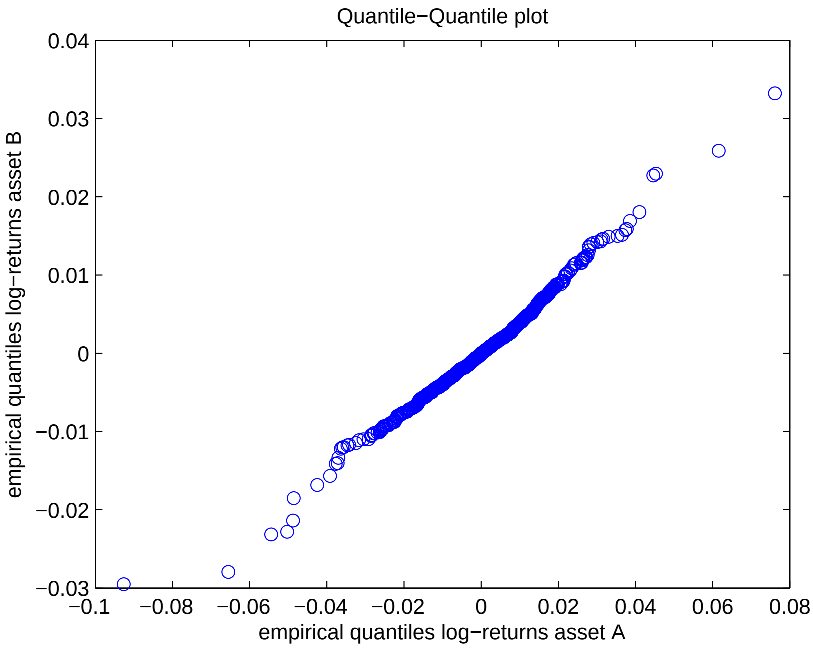

Figure 2: QQplots of the marginal distributions

SOLUTION TO EXAMINATION IN SF2980 RISK MANAGEMENT 2009-12-18.

Problem 1

A confidence interval can be constructed as $(X_{j,n}, X_{k,n})$ where $X_{1,n} \geq \cdots \geq X_{n,n}$ is the ordered sample. Then

$$P(X_{k,n} < F^{-1}(0.99)) = P(\#\{X_i > F^{-1}(0.99) \leq k-1) = P(\mathrm{Bin}(1000, 0.01) \leq k-1),$$
$$P(X_{j,n} > F^{-1}(0.99)) = P(\#\{X_i > F^{-1}(0.99) \geq i) = P(\mathrm{Bin}(1000, 0.01) \geq j).$$

For given k and j the exact confidence level is then computed as $1-P(\mathrm{Bin}(1000, 0.01) \leq k-1) - P(\mathrm{Bin}(1000, 0.01) \geq j)$.

Problem 2

In the empirical methods $F^{-1}(u)$ is replaced by the empirical quantile function $F_n^{-1}(u)$, where $F_n^{-1}(u) = X_{[n(1-u)]+1,n}$ and $X_{1,n} \geq X_{2,n} \geq \cdots \geq X_{n,n}$ is the ordered sample. With $n = 100$ see that for $0.98 < u \leq 0.99$ we have $[100(1-u)]+1 = 2$ and $F_n^{-1}(u) = X_{2,n}$ and similarly for $u \geq 0.99$, $F_n^{-1}(u) = X_{1,n}$. It follows that

$$\frac{1}{0.02}\int_{0.98}^{1} F_n^{-1}(u)du = \frac{1}{0.02}(0.01X_{2,100} + 0.01X_{1,100}) = \frac{1}{2}(X_{2,100} + X_{1,100}).$$

That is, the average of the two largest values in the sample. The advantage with the empirical approach is that it is easy to use and there is no distributional assumption. The problem is the current setting is that there are only two observations which makes the estimate very unstable and not reliable.
(b) I'll give a brief answer. In the POT method it is assumed that $\overline{F}(u)$ is regularly varying. That is $\lim_{t\to\infty} \overline{F}(tu)/\overline{F}(t) = u^{-\alpha}$, for some $\alpha > 0$. There are a number of methods to check the validity of this assumption. One example is to do a mean excess plot and see if it looks linear with positive slope above some threshold u_0. If that is the case the POT method suggest to approximate the excess distribution $P(X > u_0 + x \mid X > u_0)$ by a Generalized Pareto Distribution (GPD). The parameters of the GPD can be fitted to the excesses using maximum likelihood and then the integrated quantile function can be obtained by replacing $F^{-1}(u)$ by $F_{POT}^{-1}(u)$ where $\overline{F}_{POT}(u_0 + x) = [N_{u_0}/n]\overline{GPD}(x)$. Here N_{u_0} is the number of excesses over u_0 and $\overline{GPD}$ is the tail of the fitted GPD distribution. The good thing with the POT method is that it can provide a more stable behavior in the extreme region where there are only few data points. Some disadvantages can be that it may be difficult to confirm the validity of the regular variation assumption and the results may be sensitive to the arbitrary choice of the threshold u_0.

Problem 3

First note that the copula of (X, Y) is the distribution of $(1-U, V)$ which is given by

$$\begin{aligned}
\widetilde{C}(u,v) &= P(1-U \leq u, V \leq v) \\
&= P(U \geq 1-u, V \leq v) \\
&= P(V \leq v) - P(U \leq 1-u, V \leq v) \\
&= v - C(1-u, v).
\end{aligned}$$

(a) For the copula in (a) we have

$$\begin{aligned}\lambda_L(X,Y) &= \lim_{u\to 0}\frac{\widetilde{C}(u,u)}{u}\\ &= \lim_{u\to 0}\frac{u-((1-u)^{-1/\theta}+u^{-1/\theta}-1)^{-1/\theta}}{u}=0.\end{aligned}$$

(b) Here it is sufficient to check that (X,Y) has a joint Gaussian distribution and hence $\lambda_L = 0$. To see that (X,Y) has a Gaussian distribution, note that $\Phi^{-1}(1-u) = -\Phi^{-1}(u)$, by symmetry. Then $(X,Y) = (-\Phi^{-1}(U), \Phi^{-1}(V)) = B(\Phi^{-1}(U), \Phi^{-1}(V))$ where $B = \mathrm{diag}(-1,1)$. Since (U,V) has a Gaussian copula $(\Phi^{-1}(U), \Phi^{-1}(V))$ has a joint Gaussian distribution and so does (X,Y).

Problem 4

Based on the scatter plot and the qq-plots it may be assumed that the log-returns $\mathbf{Y} = (Y_A, Y_B)$ has a normal variance mixture distribution; that is $\mathbf{Y} \stackrel{d}{=} \boldsymbol{\mu} + WA\mathbf{Z}$ with $W \geq 0$ and $\mathbf{Z} \sim N(0,I)$, independent. Based on the plots the location vector $\boldsymbol{\mu}$ is approximated by 0. Then for any portfolio $(h_1,h_2)^{\mathrm{T}}$ investing in h_1 shares of the first asset and h_2 shares of the second asset the linearized loss can be written

$$L^{\Delta} = -\mathbf{w}^{\mathrm{T}}\mathbf{Y} \stackrel{d}{=} \mathbf{w}^T\mathbf{Y} \stackrel{d}{=} \sqrt{\mathbf{w}^T\Sigma\mathbf{w}}\,WZ_1,$$

with $\mathbf{w}^T = (w_1,w_2) = (h_1S_A, h_2S_B)$. Then the 0.95-quantile is

$$F^{-1}_{L^{\Delta}}(0.95) = \sqrt{\mathbf{w}^T\Sigma\mathbf{w}}\,F^{-1}_{WZ_1}(0.95).$$

We see that the only thing that differs for the three portfolios is the value of $\mathbf{w}^T\Sigma\mathbf{w}$. For portfolio (1)

$$\begin{aligned}\mathbf{w}^T\Sigma\mathbf{w} &= (300,300)\begin{pmatrix}\sigma_1^2 & \sigma_1\sigma_2\rho\\ \sigma_1\sigma_2\rho & \sigma_2^2\end{pmatrix}\begin{pmatrix}300\\300\end{pmatrix}\\ &= 300^2(\sigma_1^2+2\sigma_1\sigma_2\rho+\sigma_2^2).\end{aligned}$$

For portfolio (2) we get

$$\begin{aligned}\mathbf{w}^T\Sigma\mathbf{w} &= (600,0)\begin{pmatrix}\sigma_1^2 & \sigma_1\sigma_2\rho\\ \sigma_1\sigma_2\rho & \sigma_2^2\end{pmatrix}\begin{pmatrix}600\\0\end{pmatrix}\\ &= 600^2\sigma_1^2,\end{aligned}$$

and for portfolio (3)

$$\begin{aligned}\mathbf{w}^T\Sigma\mathbf{w} &= (0,600)\begin{pmatrix}\sigma_1^2 & \sigma_1\sigma_2\rho\\ \sigma_1\sigma_2\rho & \sigma_2^2\end{pmatrix}\begin{pmatrix}0\\600\end{pmatrix}\\ &= 600^2\sigma_2^2.\end{aligned}$$

To get any further we need some relation between σ_1 and σ_2. This can be obtained from the QQ-plot. Since $Y_B \stackrel{d}{=} \sigma_2WZ_1 \stackrel{d}{=} (\sigma_2/\sigma_1)\sigma_1WZ_1 \stackrel{d}{=} (\sigma_2/\sigma_1)Y_B$ we have

$F^{-1}_{Y_B}(p) = (\sigma_2/\sigma_)F^{-1}_{Y_A}(p)$. That is, we can get σ_2/σ_1 as the slope of the QQ-plot. It is approximately 0.4. Then $\sigma_2^2 \approx 0.16\sigma_1^2$ and

$$\begin{aligned}
&\text{Portfolio 1:} \quad \mathbf{w}^T\Sigma\mathbf{W} \approx 100^2\sigma_1^2 \cdot 9(1 + 2\cdot 0.4\rho + 0.16),\\
&\text{Portfolio 2:} \quad \mathbf{w}^T\Sigma\mathbf{W} \approx 100^2\sigma_1^2 \cdot 36,\\
&\text{Portfolio 3:} \quad \mathbf{w}^T\Sigma\mathbf{W} \approx 100^2\sigma_1^2 \cdot 36 \cdot 0.16 < 100^2\sigma_1^2 \cdot 6.
\end{aligned}$$

Since $\rho > 0$ seems reasonable the riskiest portfolio is Portfolio 2 and the least risky portfolio is Portfolio 3.

Problem 5

Take any sequence of 0's and 1's of length n. Then

$$\begin{aligned}
P(X_1 = 1, \dots, X_n = 0 \mid Z) &= P(Y_1 \le F^{-1}_{Y_i}(p_1), \dots, Y_n > F^{-1}_{Y_n}(p_n) \mid Z)\\
&= P(Z_1 \le Z + F^{-1}_{Y_i}(p_1), \dots, Z_n > Z + F^{-1}_{Y_n}(p_n) \mid Z)\\
&= P(Z_1 \le Z + F^{-1}_{Y_i}(p_1) \mid Z) \cdots P(Z_n > Z + F^{-1}_{Y_n}(p_n) \mid Z),
\end{aligned} \tag{1}$$

by independence of $Z, Z_1, \dots, Z_n$. Since each term of the product is of the form

$$\begin{aligned}
&P(Z_i \le Z + F^{-1}_{Y_i}(p_i) \mid Z) = P(Y_i \le F^{-1}_{Y_i}(p_i) \mid Z) = P(X_i = 1 \mid Z), \quad \text{or}\\
&P(Z_i > Z + F^{-1}_{Y_i}(p_i) \mid Z) = P(Y_i > F^{-1}_{Y_i}(p_i) \mid Z) = P(X_i = 0 \mid Z)
\end{aligned}$$

we see that (1) is equal to

$$P(X_1 = 1 \mid Z) \cdots P(X_n = 0 \mid Z).$$

This shows (a). For (b) We can write

$$\begin{aligned}
P(X_i = 1 \mid Z) &= P(Y_i \le d_i \mid Z)\\
&= P(Z_i \le Z + d_i \mid Z)\\
&= F_{Z_i}(Z + d_i).
\end{aligned}$$