KTH Matematik

SOLUTION TO EXAMINATION IN SF2980 RISK MANAGEMENT, 2008-12-19, 08:00–13:00.

Examiner: Henrik Hult, tel. 790 6911, e-mail: hult@kth.se

- -

Problem 1

Select n such that

$$0.999 \leq P(\max\{X_1, \dots, X_n\} > \mathrm{VaR}_{0.99}(F)) = 1 - 0.99^n.$$

Then $n \geq \log(0.001)/\log(0.99) = 687.3$. Hence, n has to be at least 688.

Problem 2

(a) Let $F_x^{\leftarrow}$ and $F_y^{\leftarrow}$ denote the quantiles on the x and y axis of the QQ plot. If the QQ plot curves downwards then the distance between quantiles of y is smaller than the distance between quantiles of x and hence F_x has heavier tail than F_y. Similarly if the QQ plot curves upwards then F_y has heavier tail than F_x.
From the distribution function of the GPD we see that $1-G_{\gamma,\beta}(x) = (1+\gamma x/\beta)^{-1/\gamma}$ decays slower if γ is large. Hence, higher γ corresponds to heavier tails.
From the left QQ plot we see that sample 2 has heavier tail than sample 1 and from the right QQ plot we see that it also has heavier tail than sample 3. Hence, sample 2 has heaviest tail which corresponds to $\gamma^{(2)} = 0.53$. From the middle plot we see that sample 1 has heavier tail than sample 3 so it must be that $\gamma^{(1)} = 0.28$ and $\gamma^{(3)} = 0.11$. Thus, the correct alternative is (v).
(b) The mean excess function of the GPD is given by

$$\begin{aligned} e(u) &= E[X - u \mid X > u] \\ &= \frac{1}{P(X>u)} \int_u^\infty P(X \geq x)dx \\ &= \frac{1}{(1+\gamma u/\beta)^{-1/\gamma}} \int_u^\infty (1+\gamma x/\beta)^{-1/\gamma} \\ &= \frac{1}{(1+\gamma u/\beta)^{-1/\gamma}} \frac{\beta}{1-\gamma} (1+\gamma u/\beta)^{1-1/\gamma} \\ &= \frac{\gamma}{1-\gamma} u + \frac{\beta}{1-\gamma}. \end{aligned}$$

We see that the mean excess function is linear with slope $\gamma/(1-\gamma)$. The same holds then for the POT approximation of the distribution of X by

$$P(X > x+u) = \frac{N_u}{n} \overline{G}_{\beta,\gamma}(x).$$

From the mean excess plots we can estimate the slope as approximately slope left ≈ 0.2, slope middle ≈ 1.25, and slope right ≈ 0.5. This gives $\gamma_{\text{left}} \approx 0.16$, $\gamma_{\text{middle}} \approx 0.57$, and $\gamma_{\text{right}} \approx 0.33$. So matching with the estimated values gives

- Left plot $\leftrightarrow \gamma = 0.11 \leftrightarrow$ sample 3.
- Midde plot $\leftrightarrow \gamma = 0.53 \leftrightarrow$ sample 2.
- Right plot $\leftrightarrow \gamma = 0.28 \leftrightarrow$ sample 1.

Problem 3

If $X = (X_1, X_2)$ has an elliptical distribution then X has representation $X \stackrel{\mathrm{d}}{=} \mu + RAS$ where $R \geq 0$, A is a matrix, and S is independent of R and uniformly distributed on the unit circle. In particular $X_1 \stackrel{\mathrm{d}}{=} \mu_1 + R(AS)_1$ and $X_2 \stackrel{\mathrm{d}}{=} \mu_2 + R(AS)_2$ which means that they have the same distribution up to location and scale. That is, $X_1 \stackrel{\mathrm{d}}{=} aX_2 + b$ for some constants a and b. Let F_1 and F_2 denote the marginal distributions of X_1 and X_2. Then it follows that $F_1^{\leftarrow}(\cdot) = aF_2^{\leftarrow}(\cdot) + b$. In particular, if for some reference distribution G and some constants c and d, $G^{\leftarrow}(\cdot) = cF_1^{\leftarrow}(\cdot) + d$ then it follows that $G^{\leftarrow}(\cdot) = caF_2^{\leftarrow}(\cdot) + d + b$. In other words, if the QQ plot of a sample from X_1 with respect to a reference distribution G is linear, then the QQ plot of a sample from X_2 with respect to the same G is also linear.
We see from the given QQ plots that F_1 appears to be close to a t_6 distribution because the QQ plot is linear, whereas F_2 seems close to a normal distribution. As argued above this is not compatible with (X_1, X_2) having an elliptical distribution, so the answer is NO.

Problem 4

By inverting F_1 and F_2 we find that $F_1^{-1}(u) = -\ln(1-u)$ and $F_2^{-1}(v) = -\frac{1}{2}\ln(1-u)$. Let

$$A = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix},$$

so that

$$AA^T = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

is a correlation matrix. To generate one sample we do the following.

- Generate Z_1 and Z_2 as independent $N(0,1)$.
- Put $(X_1, X_2)^T = A(Z_1, Z_2)^T$.
- Put $(U_1, U_2)^T = (\Phi(X_1), \Phi(X_2))^T$.
- Put $(Y_1, Y_2)^T = (F_1^{-1}(U_1), F_2^{-1}(U_2))^T$.

Then $(Y_1, Y_2)^T$ has the desired distribution. Indeed,

$$\begin{aligned}P(Y_1 \leq y_1, Y_2 \leq y_2) &= P(F_1^{-1}(U_1) \leq y_1, F_2^{-1}(U_2) \leq y_2) \\ &= P(U_1 \leq F_1(y_1), U_2 \leq F_2(y_2)) \\ &= C(F_1(y_1), F_2(y_2)),\end{aligned}$$

where C is the Gaussian copula given by

$$C(u_1, u_2) = \Phi(\Phi^{-1}(u_1), \Phi^{-1}(u_2)).$$

Problem 5

First note that $P(X_i = 1) = p = 1 - P(X_i = 0)$ and the same for X_j. Then $E[X_i] = p$ and $\mathrm{var}(X_i) = p(1-p)$ and the same for X_j. We will also use that $p = P(Y_j \leq F_j^{\leftarrow}(p))$. Write

$$\begin{aligned}\rho_L(X_i, X_j) &= \frac{\mathrm{Cov}(X_i, X_j)}{\sqrt{\mathrm{var}(X_i)\,\mathrm{var}(X_j)}} \\ &= \frac{E[X_i X_j] - E[X_i]E[X_j]}{\sqrt{p(1-p)p(1-p)}} \\ &= \frac{P(X_i = 1, X_j = 1) - p^2}{p(1-p)} \\ &= \frac{P(Y_i \leq F_i^{\leftarrow}(p), Y_j \leq F_j^{\leftarrow}(p)) - p^2}{p(1-p)} \\ &= \frac{P(Y_i \leq F_i^{\leftarrow}(p), Y_j \leq F_j^{\leftarrow}(p))}{P(Y_j \leq F_j^{\leftarrow}(p))(1-p)} - \frac{p^2}{p(1-p)} \\ &= \frac{P(Y_i \leq F_i^{\leftarrow}(p) \mid Y_j \leq F_j^{\leftarrow}(p))}{1-p} - \frac{p^2}{p(1-p)}.\end{aligned}$$

Now, since $\lambda_L(Y_i, Y_j) = \lim_{p\to 0} P(Y_i \leq F_i^{\leftarrow}(p) \mid Y_j \leq F_j^{\leftarrow}(p))$ by definition, it follows that

$$\lim_{p\to 0} \rho_L(X_i, X_j) = \lim_{p\to 0} \frac{P(Y_i \leq F_i^{\leftarrow}(p) \mid Y_j \leq F_j^{\leftarrow}(p))}{1-p} - \frac{p^2}{p(1-p)} = \lambda_L(Y_i, Y_j).$$