Black-Scholes model with continuous dividend yield

Assume that $\underline{\mathcal{F}} = \underline{\mathcal{F}}^W$

The S-market under P:

$$\begin{cases} dB_t &= rB_t dt \\ dS_t &= \alpha S_t dt + \sigma S_t dW_t \\ dD_t &= \delta S_t dt + \gamma S_t dW_t \end{cases} \tag{1}$$

where α, β, σ, δ, and γ are constants.

We want to find a martingale measure Q, i.e. a probability measure Q such that

- $Q \sim P$,
- G^Z is a Q-martingale.

The Girsanov transformation:

Define Q by

$$dQ = L_T dP \quad \text{on } \mathcal{F}_T, \tag{2}$$

where

$$\begin{cases} dL_t &= g_t L_t dW_t, \\ L_0 &= 1. \end{cases} \tag{3}$$

The Girsanov theorem $\Longrightarrow$

$$dW_t = g_t dt + dV_t,$$

where V is a Q-Wiener process.

The S-market under Q:

$$\begin{cases} dS_t &= (\alpha + \sigma g_t) S_t dt + \sigma S_t dV_t \\ dD_t &= (\delta + \gamma g_t) S_t dt + \gamma S_t dV_t \end{cases}$$

The Q-dynamics of the Z-economy

$$[Z_t, D_t^Z] = \left[\frac{S_t}{B_t}, \int_0^t \frac{1}{B_s} dD_s \right]$$

are given by

$$\begin{cases} dZ_t &= (\alpha - r + \sigma g_t) Z_t dt + \sigma Z_t dV_t \\ dD_t^Z &= (\delta + \gamma g_t) Z_t dt + \gamma Z_t dV_t \end{cases}$$

The Q-dynamics of G^Z are thus

$$
\begin{aligned}
dG_t^Z &= dZ_t + dD_t^Z \\
&= (\alpha - r + \sigma g_t + \delta + \gamma g_t) Z_t dt \\
&\quad + (\sigma + \gamma) Z_t dV_t.
\end{aligned}
$$

Now, G^Z is a Q-martingale if

$$
g = \frac{r - \alpha - \delta}{\sigma + \gamma}. \tag{4}
$$

Prop. For the S-market (1) it holds that

1. The market is free of arbitrage and complete since there exists a unique martingale measure given by (2), (3), and (4).

2. The arbitrage free price of a T-claim X is

   $$\Pi_t(X) = E^Q[X|\mathcal{F}_t]$$

   (use that $G^Z = \Pi$ is a Q-MG) where the Q-dynamics of S and D are

   $$\begin{cases} dS_t &= (\alpha + \sigma\frac{r-\alpha-\delta}{\sigma+\gamma})S_t dt + \sigma S_t dV_t \\ dD_t &= (\delta + \gamma\frac{r-\alpha-\delta}{\sigma+\gamma})S_t dt + \gamma S_t dV_t \end{cases}$$

   Note: The local rate of return of the stock is **not** equal to r!

3. If $\gamma = 0$ then the Q-dynamics of $[S, D]$ are

$$\begin{cases} dS_t & = & (r - \delta) S_t dt + \sigma S_t dV_t \\ dD_t & = & \delta S_t dt \end{cases}$$

Give economic interpretation!

4. If $\gamma = 0$ and $X = \phi(S_T)$ then the arbitrage free price of X is given by

$$\Pi_t(X) = F(t, S_t)$$

where F solves the PDE

$$\begin{cases} F_t + (r - \delta) s F_s + \frac{1}{2}\sigma^2 s^2 F_{ss} - rF & = & 0 \\ F(T, s) = \phi(s) \end{cases}$$

Read about a more general model in 16.2.2