LECTURE 9
CH. 6.2-6.3 FROM A. FRIEDMAN

Theorem. (AFr 6.2.4.) (Riesz theorem)
Let H be a Hilbert space. For any bounded linear fuctional $x^* : H \to \mathbb{C}$, there exists $z \in H$ s.t.

- $x^*(x) = (x, z), \qquad x \in H.$
- $\|x^*\| = \|z\|.$

Definition.
Let H be a Hilbert space and let $M \subset H$ be a closed linear subspace. Then according to Theorem 6.2.2 for any $x \in H$ there exist unique vectors $y, z \in H$ such that

$$x = y + z, \qquad y \in M, \quad z \text{ orthogonal to } M.$$

We say then that y is the projection of x on M and define the linear operator $P : H \to M$ such that $Px = y$. P is called the projection operator on M.

Definition.
Let T be a bounded linear operator $T : H \to H$. The adjoint T^* of T is defined by the equality

$$(Tx, y) = (x, T^*y), \qquad \forall x, y \in H.$$

If $T = T^*$ then T is called self-adjoint.

Remark. If T is self-adjoint then the scalar product (Tx, y) is real.

Theorem. (AFr 6.3.1.)
Let P be a projection. Then

- P is a self-adjoint linear operator.
- $P^2 = P$.
- $\|P\| = 1$ if $P \neq 0$.

Theorem. (AFr 6.3.2.)
If P is a self-adjoint linear operator s.t. $P^2 = P$, then P is a projection.

Definition.
Let P_1 and P_2 be projections in H. We say that P_1 is orthogonal to P_2 is $P_1 P_2 = 0$.

Remark. Since $(P_1P_2)^* = P_2^*P_1^* = P_2P_1$, then $P_1P_2 = 0$ implies $P_2P_1 = 0$.

Theorem. (AFr 6.3.4.)
The operator $P_1 + P_2$ is a projection *iff* $P_1P_2 = 0$.

Theorem. (AFr 6.3.5.)
The product $P_1P_2 = 0$ is a projection *iff* $P_1P_2 = P_2P_1$.

Home exercises.

1. *(ex. 6.3.3 from AFr)* Let a linear operator P satisfies the properties $P^* = P$ and P^2 is a projection. Is P a projection?

2. *(ex. 6.3.5 from AFr)* Confider the operator $Qf(t) = a(t)f(t)$in $L^2(0,1)$, where $a(t)$ is a scalar function. Find necessary and sufficient conditions on $a(t)$ for Q to be a projection.

3. Let $H = L^2(-\infty, \infty)$ and let

$$\chi(x) = \begin{cases} 1, & |x| < 1, \\ 0, & |x| \geq 1 \end{cases}.$$

Show that

- the operator $Pf(x) = \chi(x)f(x)$ is a projection
- Let $\mathcal{F}$ be the Fourier transform
  $$\mathcal{F}f(\xi) = \int_{-\infty}^{\infty} f(x)e^{-ix\xi}\,dx.$$
  Show that the operator Q defined by
  $$Q\,f = \mathcal{F}^{-1}\chi\mathcal{F}\,f$$
  is a projection.
- Is PQP a projection?