LECTURE 8
CH. 6 FROM A. FRIEDMAN

Definition.
H is called a Hilbert space if H is a complex linear space supplied with $(\cdot,\cdot): H \times H \to \mathbb{C}$ s.t.

- $(x, x) \geq 0$, & $(x, x) = 0 \Leftrightarrow x = 0$.
- $(x + y, z) = (x, z) + (y, z)$, $\forall x, y, z \in H$.
- $(\lambda x, y) = \lambda(x, y)$, $\forall x, y \in H$, $\lambda \in \mathbb{C}$.
- $(x, y) = \overline{(y, x)}$, $\forall x, y \in H$.
- If $\{x_n\}$ is a Cauchy sequence and $\lim_{n,m\to\infty}(x_n - x_m, x_n - x_m) = 0$, then there exists $x \in H$, s.t. $\lim_{n\to\infty}(x_n - x, x_n - x) = 0$.

$(\cdot,\cdot)$ is called scalar product.
$\|x\| = \sqrt{(x, x)}$ is called the norm of x.

Examples.

- $H = l^2 = \{\mathbf{a} = \{a_n\}_{n=1}^{\infty}\}$, such that $\sum_{n=1}^{\infty} |a_n|^2 < \infty$. We define scalar product
$$(\mathbf{a}, \mathbf{b}) = \sum_{n=1}^{\infty} a_n \overline{b_n}, \qquad \mathbf{a}, \mathbf{b} \in H.$$
- $H = L^2(0, 1) = \{f : \int_0^1 |f(x)|^2\, dx < \infty\}$ with scalar product
$$(f, g) = \int_0^1 f(x)\overline{g(x)}\, dx.$$
- Sobolev space
$$H^1(0, 1) = \left\{ f : \int_0^1 \left(|f'(x)|^2 + |f(x)|^2 \right) dx < \infty \right\}.$$
The corresponding scalar product is equal to
$$(f, g) = \int_0^1 \left(f'(x)\overline{g(x)} + f(x)\overline{g(x)} \right) dx.$$

Theorem. (AFr 6.1.1) (Schwartz inequality)
Let H be a Hilbert space, $x, y \in H$. Then
$$|(x, y)| \leq \|x\|\|y\|.$$

Theorem. (AFr 6.1.2)
Any Hilbert space is a Banach space whose norm is equal to $\|\cdot\| = \sqrt{(\cdot,\cdot)}$.

Corollary. (AFr 6.1.3)
The norm in a Hilbert space is strictly convex, namely

$$\|x\| = \|y\| = 1, \quad \|x+y\| = 2 \quad \Rightarrow \quad x = y.$$

Theorem. (AFr 6.1.4)
Let H be a Hilbert space. Then its norm satisfies the identity

$$2\Big(\|x\|^2 + \|y\|^2\Big) = \|x+y\|^2 + \|x-y\|^2.$$

Theorem. (AFr 6.1.5)
If H is a Banach space with norm satisfying Th 6.1.4 then H is a Hilbert space.

Definition. Let H be a Hilbert space, $x, y \in H$. We say that x is orthogonal to y if $(x,y) = 0$.

Let $M \subset H$. We say that x is orthogonal to M if $(x,y) = 0$, $\forall y \in M$.

Let $N, M \subset H$. We say that N is orthogonal to M if $(x,y) = 0$, $x \in N$, $y \in M$.

Lemma. (AFr 6.2.1)
Let H be a Hilbert space and let $M \subset H$ be a closed convex subset. Then for any x_0 there exists unique element $y_0 \in M$ s.t.

$$\|x_0 - y_0\| = \inf_{y \in M} \|x_0 - y\|.$$

Theorem. (AFr 6.2.2)
Let M be a closed subspace of a Hilbert space H. Then for any $x_0 \in H$ there are elements $y_0 \in M$ and z_0 orthogonal to M s.t.

$$x_0 = y_0 + z_0$$

and this decomposition is unique.

Home exercises.

Let A be a 2×2 real matrix. Define

$$|A| = \sqrt{A^T A}.$$

We say that $A \leq B$, if for any vector X

$$X^T A X \leq X^T B X.$$

1. Let

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Show that

$$\left|(\sigma_3 + I) + (\sigma_1 - I)\right| \not\leq \left|(\sigma_3 + I)\right| + \left|(\sigma_1 - I)\right|.$$

2. Let A and B be two 2×2 real matrices. Show that the inequality

$$\| \, |A| - |B| \, \| \leq \|A - B\|$$

is not always true.