LECTURE 7
CH. 4.8 FROM A. FRIEDMAN
FOUNDATIONS OF MORDERN ANALYSIS

Theorem. (Hahn-Banach) (AFr 4.8.2)
Let X be a normed linear vector space and let $Y \subset X$ be a linear subspace. Then for any $y^* \in Y^*$ there exists $x^* \in X^*$ s.t.

$$\|x^*\| = \|y^*\| \qquad \& \qquad x^*(y) = y^*(y), \quad \forall y \in Y.$$

Theorem. (AFr 4.8.3)
Let X be a normed linear vector space and let $Y \subset X$ be a linear subspace. Let $x_0 \in X$ s.t.

$$\inf_{y \in Y} \|y - x_0\| = d > 0.$$

Then there exists $x^* \in X^*$ s.t.

$$x^*(x_0) = 1, \qquad \|x^*\| = \frac{1}{d} \quad \text{and} \quad x^*(y) = 0, \quad \forall y \in Y.$$

Corollary. (AFr 4.8.4)
If X is a normed linear space, then for any $x \neq 0$ there exists $x^* \in X^*$ s.t. $\|x^*\| = 1$ and $x^*(x) = \|x\|$.

Corollary. (AFr 4.8.5)
If X is a normed linear space and if $y \neq z$, $y, z \in X$, then there exists $x^* \in X^*$ s.t. $x^*(y) \neq x^*(z)$.

Corollary. (AFr 4.8.6)
Let X be a normed linear vector space. Then for any $x \in X$

$$\|x\| = \sup_{x^* \neq 0} \frac{|x^*(x)|}{\|x^*\|} = \sup_{\|x^*\|=1} |x^*(x)|.$$

Corollary. (AFr 4.8.7)
Let X be a normed linear vector space and let $Y \subset X$ be a linear subspace. Assume that $\overline{Y} \neq X$. Then there exists $x^* \neq 0$ s.t. $x^*(y) = 0$, $\forall y \in Y$.

Definition. The null space of $x^* \in X^*$ is the set

$$N_{x^*} = \{x \in X : \; x^*(x) = 0\}.$$

Let $x^* \neq 0$. Then there is $x_0 \neq 0$ s.t. $x^*(x) = 1$ and any $x \in X$ can be written as $x = z + \lambda x_0$, where $\lambda = x^*(x)$ and $z = x - \lambda x_0 \in N_{x^*}$.

Example. Let $f \in L^p(0,1)$ $g \in L^q(0,1)$ $1/p + 1/q = 1$ be real functions and define a linear functional G^* on $L^p(0,1)$ such that

$$G^*(f) = \int_0^1 f(x)g(x)\,dx.$$

Then $N_g = \{f \in L^p(0,1) : \int f(x)g(x)\,dx = 0\}$.

Definition. Let $c \in \mathbb{R}$, $x^* \neq 0$. The set

$$\{x \in X : \operatorname{Re} x^*(x) = c\}$$

is called hyperplane.
The sets $\{x \in X : \operatorname{Re} x^*(x) \geq c\}$, $\{x \in X : \operatorname{Re} x^*(x) \leq c\}$ are called half-spaces.
Any hyperplane coincides with $N_{x^*} + cx_0$, $x^*(x_0) = 1$ for some x^*, x_0 and $c \in \mathbb{R}$.

Definition. Let $K \subset X$ be a subset of a normed linear space X and let $x_0 \in K$. If for $x^* \in X^*$ we have $\operatorname{Re} x^*(x) \leq \operatorname{Re} x^*(x_0)$ for any $x_0 \in K$, then we say that x^* supports K at x_0 (or that x^* is tangent to Kat x_0).
The hyperplane $\{x \in X : \operatorname{Re} x^*(x) = \|x_0\|\}$ is called a supporting (or tangent) hyperplane to K at x_0.

Corollary. (AFr 4.8.8)
Let X be a normed linear space and let $x_0 \in \overline{B} = \{x : \|x\| \leq 1\}$. Then for any x_0 s.t. $\|x_0\| = 1$ there is a tangent hyperplane to B at x_0.

Home exercises.

1. Define C_0 as a set of sequences $\{a_k\}_{k=1}^{\infty}$ for which $\lim_{k\to\infty} a_k = 0$. If we introduce the following norm

$$\|\{a_k\}_{k=1}^{\infty}\| = \max_{k\in\mathbb{N}} |a_k|,$$

then C_0 becomes a normed linear space.
Assume that $\{\lambda\}_{k=1}^{\infty} \in l_1$ $\left(\Leftrightarrow \sum_{k=1}^{\infty} |\lambda_k| \leq \infty \right)$.
Show that

$$\Lambda(\{a_k\}_{k=1}^{\infty}) = \sum_{k=1}^{\infty} \lambda_k a_k$$

is a linear functional on C_0 and $\|\Lambda\| = \sum_{k=1}^{\infty} |\lambda_k|$.

2. Show that $l_\infty = l_1^*$ but $l_\infty^* \neq l_1$.

3. Let X be a Banach space. We say that the functional φ is convex if

$$\varphi\Big(\frac{x+y}{2}\Big) \leq \frac{1}{2}\Big(\varphi(x)+\varphi(y)\Big).$$

Define a mapping $L:\ X \to X^*$, s.t.

$$L\varphi(x^*) = \psi(x^*) = \sup_{x\in X}\Big(x^*(x)-\varphi(x)\Big).$$

Show that

- ψ is convex.
- If now $L^*:\ X^* \to X$ is defined by

  $$L^*\varphi(x) = \sup_{x^*\in X^*}\Big(x^*(x)-\psi(x^*)\Big),$$

  then $L^*L\varphi = \varphi$.