LECTURE 6
CH. 4.7 - 4.8 FROM A. FRIEDMAN
FOUNDATIONS OF MORDERN ANALYSIS

Example 1. Let $X = L^2(0,1)$ and let $Tf(x) = f(0)$ with $D_T = C^1[0,1]$.
We show that T cannot be closed.
Indeed, let $f_n(x) = (1-x)^n$. Then $f_n \in D_T$, $\|f_n\|_{L^2(0,1)} \to 0$, $t \to \infty$, but $Tf_n = 1$.
Note that in this case $G_T = \{(f, f(0)) : f \in D_T\}$ and $\|(f, Tf)\| = \|f\|_{L^2} + |f(0)|$.

Example 2. Let $X = L^p(0,1)$, $p > 1$, $Tf(x) = f'(x)$ and $D_T = C^1[0,1]$.
We show that T can be closed.
The operator T is not closed as it is defined on $C^1[0,1]$ which is not a closed set in $L^p(0,1)$. In order to show that T can be closed if is enough to show that

$$f_n \in D_T, \quad f_n \to 0, \quad Tf_n \to g \quad \Longrightarrow \quad g = 0.$$

Let $\|f_n\|_{L^p} \to 0$, $f_n \in C^1[0,1]$ and let

$$\int_0^1 |f_n'(x) - g(x)|^p dx \to 0.$$

If $1/p + 1/q = 1$, then for any $\varphi \in C_0^\infty(0,1)$ we have

$$\Big|\int_0^1 (f_n'(x) - g(x))\varphi(x)\,dx\Big| \le \Big(\int_0^1 |f_n'(x) - g(x)|^p\,dx\Big)^{1/p}\Big(\int_0^1 \varphi(x)|^q\,dx\Big)^{1/q} \to 0,$$

which implies

$$\int_0^1 f_n'(x)\varphi(x)\,dx \to \int_0^1 g(x)\varphi(x)\,dx.$$

On the other hand

$$\Big|\int_0^1 f_n'(x)\varphi(x)\,dx\Big| \le \Big(\int_0^1 |f_n'(x))|^p\,dx\Big)^{1/p}\Big(\int_0^1 \varphi(x)|^q\,dx\Big)^{1/q} \to 0, \qquad n \to \infty.$$

Thus

$$\Big|\int_0^1 g(x)\varphi(x)\,dx\Big| = 0$$

for any $\varphi \in C_0^\infty(0,1)$ which means that $g = 0$ as an L^p function.

Definition. Let X be a linear vector space and let $p : X \to \mathbb{R}(\mathbb{C})$ be a linear operator. Then p is called a real(complex) linear functional.

Examples.

- $p : f(x) \to f(0), f \in C(-1, 1)$.
- Let $f \in L^(0, 2\pi)$ and let
$$f(x) = \sum_{-\infty}^{\infty} a_n e^{-inx}/\sqrt{2\pi}$$
be its Fourier series. Then $p \to a_n$ is a complex linear functional on $L^2(0, 2\pi)$.
- Let $g \in L^2(0, 1)$. We define $p = p_g$ such that
$$p_g(f) = \int_0^1 f(x)g(x)\,dx.$$

Definition. A partially ordered set S is a non-empty set with a relation " $\leq$ " satisfying the properties:

- $x \leq x$
- $x \leq y, y \leq z \Longrightarrow x \leq z$.

If for any $x, y \in S$ either $x \leq y$ or $y \leq x$ then S is called totally ordered.

Definition. Let $T \subset S$, and S be partially ordered. The element $x \in S$ is called an upper bound if for any $y \in S$, we have $y \leq x$.

Definition. Let S be partially ordered. The element $x \in S$ is called maximal if for any $y \in S$ the relation $x \leq y$ implies $y \leq x$.

Theorem. (Zorn's lemma)
If S is a partially ordered set s.t. every totally ordered subset has an upper bound, then S has a maximal element.

Notation.
By X^* we denote the set of all continuous linear functionals on X.

Theorem. (Hahn-Banach lemma, AFr Th 4.8.1.)
Let X be a real linear vector space space and let p be a real functional (not necessary linear) on X s.t.
$$p(x + y) \leq p(x) + p(y) \qquad p(\lambda x) = \lambda p(x), \qquad \lambda > 0, \quad x, y \in X.$$
Let f be a real linear functional on a linear subspace $Y \subset X$ s.t.
$$f(x) \leq p(x), \qquad \forall x \in Y.$$

Then there exists a real linear functional F on X s.t.

$$F(x) = f(x), \qquad x \in X, \qquad \text{and} \qquad F(x) \le p(x), \quad \forall x \in X.$$

Home exercises.

1. *(ex. 4.7.6 from AFr)*
Prove that d/dx cannot be extended into a linear operator from $L^p(0,1)$ into itself.

2. *(ex. 4.7.7 from AFr)*
Let $C^m(\Omega)$ be a normed space equipped with

$$\|u\|_m = \sup_{\Omega} \sum_{0 \le |\alpha| \le m} |D^\alpha u|,$$

where $\Omega \subset \mathbb{R}^n$. Define $C^*(\Omega)$ the subset of $C^\infty(\Omega)$ of all functions u s.t. $\|u\|_m < \infty \ \forall \ m \ge 1$. Introduce in $C^*(\Omega)$ a metric $\rho(u,v) = \rho(u-v,0)$, where

$$\rho(u,0) = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{\|u\|_m}{1 + \|u\|_m}.$$

Determine if $d/dx : \ C^*(\Omega) \to C^*(\Omega)$ is continuous.

3. Let $X = L^2(0,2\pi)$, $\varphi_n = \{e^{inx}/\sqrt{2\pi}\}_{n=-\infty}^{\infty}$. Let $\psi \in X$ be an element which is not a finite linear combination of φ_n and let D be a set of all finite linear conbinations of $\{\varphi_n\}$. Denote by T the operator

$$T\Big(b\psi + \sum_{i=-M}^{N} c_j \varphi_j\Big) = b\psi.$$

Show that $\overline{G}_T$ is not a gragh. (Namely, show that $(\psi,\psi) \in \overline{G}_T$ and $(\psi, 0) \in \overline{G}_T$.)