LECTURE 5
LINEAR TRANSFORMATIONS
CH. 4.6 - 4.7 FROM AVNER FRIEDMAN
FOUNDATIONS OF MORDERN ANALYSIS

Definition. Let T be a one-to-one linear operator $T : X \to Y$. Then the inverse operator T^{-1} is defined as

$$T^{-1}y = x \qquad \text{if} \qquad Tx = y.$$

Note that domain T^{-1} is a linear subspace of Y.

Theorem. (AFr 4.6.2)
Let X and Y be a Banach spaces and let T be a one-to-one bounded linear map from X onto Y. Then T^{-1} is a bounded map.

Corollary. (AFr 4.6.3)
Let X be Banach spaces equipped with either $\|\cdot\|_1$ or $\|\cdot\|_2$. Suppose that there exists a constant K s.t.

$$\|x\|_1 \leq K\,\|x\|_2 \qquad \text{for all} \qquad x \in X.$$

Then there exists a constant K' s.t.

$$\|x\|_2 \leq K'\,\|x\|_1 \qquad \text{for all} \qquad x \in X.$$

Definition. Let X and Y be linear vector spaces and let $T : X \to Y$, where T is defined on D_T. Then

$$G_T = \{(x, Tx) : x \in D_T\}$$

is called the graph of T.
If G_T is a closed set in $X \times Y$ then we say that T is a closed operator.

Note that T is closed *iff*

$$x_n \in D_T \quad x_n \to x, \quad Tx_n \to y \Longrightarrow x \in D_T, \quad Tx = y.$$

Theorem. (AFr 4.6.4) (Closed graph theorem)
Let X and Y be Banach spaces and let $T : X \to Y$ be a linear operator with $D_T = X$. If T is closed then T is continuous.

Definition. Let X and Y be linear vector spaces and let $T: X \rightarrow Y$ defined on D_T. A linear operator $S: X \rightarrow Y$ is called an extension of T if $D_T \subset D_S$ and $Tx = Sx$ for all $x \in D_T$.

Definition. Let X be a linear vector space and Y be a normed space. Let $T: X \rightarrow Y$ defined on D_T. Suppose that there exists S such that
(i) S is a closed linear operator
(ii) S is an extension of T
(iii) If S' satisfies (i) and (ii), then S' is an extension of S.
Then we say that S is a s closure of T. The closure of T is denoted by $\overline{T}$.

Theorem. (AFr 4.7.1)
Let X and Y be Banach spaces. T has a closure $\overline{T}$ *iff*

$$x_n \in D_T \quad x_n \rightarrow 0, \quad Tx_n \rightarrow y \Longrightarrow y = 0.$$

Home exercises.

1. *(ex. 4.6.6 from AFr)*

2. *(ex. 4.6.7 from AFr)*