LECTURE 4
LINEAR TRANSFORMATIONS
CH. 4.5 - 4.6 FROM AVNER FRIEDMAN
FOUNDATIONS OF MORDERN ANALYSIS

Theorem. (AFr 4.5.1) (Banach-Steinhaus theorem)
Let X be a Banach space and Y be a normed linear space. Let $\{T_\alpha\}$ be a family of bounded linear operators from X to Y. It for each $x \in X$ the set $\{T_\alpha x\}$ is bounded, then the set $\{\|T_\alpha\|\}$ is bounded.

Definition. Let X and Y be normed linear spaces and let $T_n : X \longrightarrow Y$. The sequence $\{T_n\}_{n=1}^{\infty}$ is said to be strongly convergent if for any $x \in X$ the limit $\lim_{n\to\infty} T_n x$ exists for any $x \in X$.

If there exists a bounded T s.t. $\lim_{n\to\infty} T_n x = Tx$ for any $x \in X$, then $\{T_n\}_{n=1}^{\infty}$ is called strongly convergent to T ($T_n \longrightarrow T$).

Theorem. (AFr 4.5.2)
Let X be a Banach space and Y be a normed linear space and let $\{T_n\}_{n=1}^{\infty}$ be the sequence of bounded operators. If the sequence $\{T_n\}_{n=1}^{\infty}$ strongly convergent, then there exists T s.t. $T_n \longrightarrow T$ strongly.

Theorem. (AFr 4.6.1) (Open-mapping theorem)
Let X and Y be Banach spaces and let $T : X \longrightarrow Y$ be a mapping *onto*. Then Tmaps open sets of X onto open sets of Y.

Home exercises.

1. *(ex. 4.6.2 from AFr)*
Let X be a Banach space and let $A \in \mathcal{B}(X)$, $\|A\| < 1$. Show that $(I + A)^{-1}$ exists and

$$(I + A)^{-1} = \sum_{n=0}^{\infty} (-1)^n A^n.$$

2. *(ex. 4.6.3 from AFr)*
Let X be a Banach space and let T and T^{-1} belong to $\mathcal{B}(X)$. Show that is $S\mathcal{B}(X)$ and $\|S - T\| < 1/\|T^{-1}\|$, then S^{-1} exists and

$$\|S^{-1} - T^{-1}\| < \frac{\|T^{-1}\|}{1 - \|S - T\|\|T^{-1}\|}.$$

3. (Holmgren) Let

$$Kf(x) = \int_0^1 K(x,y)f(y)\,dy, \qquad x \in (0,1).$$

Assume that $K \in L^2(0,1) \to L^2(0,1)$ is bounded. Then

$$\|K\| \le \left(\sup_y \int_0^1 |K(x,y)|\,dx\right)^{1/2} \left(\sup_x \int_0^1 |K(x,y)|\,dy\right)^{1/2}.$$