LECTURE 3
LINEAR TRANSFORMATIONS
CH. 4.4 FROM AVNER FRIEDMAN
FOUNDATIONS OF MORDERN ANALYSIS

Definition. Linear operator.

Theorem. (AFr 4.4.1)
Let X and Y be normed linear spaces. A linear transformation $T : X \to Y$ is continuous *iff* T is continuous at one point.

Definition. Let X and Y be normed linear spaces, $T : X \to Y$ and let there exists a constant $K > 0$ s.t.

$$\|Tx\| \leq K\|x\| \qquad x \in X.$$

Then T is called a bounded linear map.

$$\|T\| = \sup_{x \in X} \frac{\|Tx\|}{\|x\|}.$$

Theorem. (AFr 4.4.2)
Let X, Y be normed linear spaces. The operator $T : X \to Y$ is continuous *iff* T is bounded.

Definition. By $\mathcal{L}(X, Y)$ we denote the space of all linear transformations equipped with

a. $T + S$

and

b. λT.

By $\mathcal{B}(X, Y) \subset \mathcal{L}(X, Y)$ we denote the set of all bounded linear transformations.

Theorem. (AFr 4.4.3)
Let X and Y be normed linear spaces. Then $\mathcal{B}(X, Y)$ is a normed linear space with the norm

$$\|T\| = \sup_{x \in X} \frac{\|Tx\|}{\|x\|}.$$

Definition. The sequence $\{T_n\}_{n=1}^{\infty}$ of bounded operators is said to be uniformly convergent if there exists bounded T s.t. $\|T_n - T\| \to 0$ as $n \to \infty$.

Theorem. (AFr 4.4.4)
If X is a normed linear space and Y is a Banach space then $\mathcal{B}(X, Y)$ is a Banach space.

Home exercises.

1. *(ex. 4.4.4 from AFr)*
Let X be a Banach space and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function. Prove that for every $T \in \mathcal{B}(X, X)$

$$\sum_{n=0}^{\infty} a_n T^n$$

is absolutely convergent in $\mathcal{B}(X) := \mathcal{B}(X, X)$.

2. *(ex. 4.4.6 from AFr)*
Find the norm of the operator $A \in \mathcal{B}(X)$given by

$$Af(t) = tf(t), \qquad 0 \le t \le 1,$$

where **a.** $X = C[0, 1]$, **b.** $X = L^p[0, 1]$, $1 \le p \le \infty$.

3. *(ex. 4.4.9 from AFr)*
Let

$$Af(x) = \int_0^1 K(x, y) f(y)\, dy, \qquad x \in (0, 1).$$

Prove that if $K \in L^2((0, 1) \times (0, 1))$ then $A : L^2(0, 1) \to L^2(0, 1)$ is bounded.

4. Let $\mathcal{L}$ be the Laplace transform defined by

$$g(s) = \mathcal{L}f(s) = \int_0^{\infty} f(t) e^{-st}\, dt.$$

Show that $\mathcal{L} : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ is bounded and

$$\|\mathcal{L}\| \le \sqrt{\pi}.$$

Tip: Write

$$\int_0^{\infty} f(t) e^{-st}\, dt = \int_0^{\infty} f(t) t^{1/2} e^{-st/2}\, t^{-1/2} e^{-st/2}\, dt$$

and use Cauchy-Schwatz inequality.