LECTURE 2
CH.4.1 FROM AVNER FRIEDMAN
FOUNDATIONS OF MORDERN ANALYSIS

Definition. Linear subspace.

Lemma. (AFr 4.3.1)
Let $Y \subset X$ be a closed linear subspace of a normed linear space X. Then for any $\varepsilon > 0$ there exists $z \in X$ s.t. $\|z\| = 1$ and $\|z - y\| > 1 - \varepsilon$, for any $y \in Y$.

Theorem. (AFr 4.3.2)
If Y is a finite-dimensional linear subspace of a normed linear space, then Y is closed.

Theorem. (AFr 4.3.3)
A normed linear space is of finite dimension **iff** every bounded subset is relatively compact.

Example. Let $X = L^2(0, 2\pi)$ and let $e_k(x) = e^{ikx}/\sqrt{2\pi}$, $k = 0, \pm 1, \dots$. Then

$$\int_0^{2\pi} |e_k(x)|^2 dx = 1 \qquad \text{and} \qquad \int_0^{2\pi} |e_k - e_j|^2 dx = 2.$$

The set $\{e_k\}_{k \in \mathbb{Z}}$ is not relatively compact.

Home exercises.
1. *(ex. 4.3.4 from AFr)*
A norm $\|\cdot\|$ is called strictly convex if $\|x\| = \|y\| = 1$ and $\|x + y\| = 2$ implies $x = y$.
Show that $L^p(\mathbb{R}^n)$ is strictly convex if $1 < p < \infty$ and is not strictly if $p = 1$ or $p = \infty$.

2. *(ex. 4.3.5 from AFr)*
Prove that $C[a, b]$ is not equivalent to L^p norm if $1 \leq p < \infty$.