LECTURE 15
MULTIPLICATION BY INDEPENDENT VARIABLE, SPECTRAL THEOREM

1. Multiplication by Independent variable

Let μ be a measure defined on σ-algebra of subsets from $\mathbb{R}$ (for definition see Av.Fr., Definition 1.2.1). Let A_μ be the operator of multiplication by independent variable in $L^2(\mathbb{R},\mu)$

$$A_\mu x(t) = tx(t), \qquad D(A_\mu) = \{x : \int_{\mathbb{R}} (1+t^2)|x(t)^2\, d\mu < \infty\}.$$

Proposition 1. *The operator A_μ is self-adjoint.*

Proof. Indeed, $\overline{D(A_\mu)} = H$ and A_μ is symmetric. Moreover

$$(A_\mu x, y) = (x, h)$$

implies $ty(t) = h(t) \in L^2(\mathbb{R},\mu)$ and $y \in D(A_\mu)$. Therefore $A_\mu^* = A_\mu$. □

Recall that we say that $\lambda \in \operatorname{supp}\mu$ if and only if $\mu(\lambda-\delta,\lambda+\delta) > 0$ for any $\delta > 0$.

Theorem 1. *The spectrum $\sigma(A_\mu)$ of the operator A_μ coincides with* supp μ *and, equivalently* $\rho(A_\mu) = \mathbb{R} \setminus \operatorname{supp}\mu$.

Proof. For any $\lambda \in \mathbb{R}$ consider $\Delta_\varepsilon = (\lambda-\varepsilon,\lambda+\varepsilon$ and suppose that $\lambda \notin \operatorname{supp}\mu$. Then for some $\varepsilon > 0$ and $x \in D(A_\mu)$ we have

$$\|A_\mu x - \lambda x\|^2 = \int_{\mathbb{R}} (t-\lambda)^2|x(t)|^2\, d\mu(t) = \int_{\mathbb{R}\setminus\Delta_\varepsilon} (t-\lambda)^2|x(t)|^2\, d\mu(t)$$
$$\geq \varepsilon^2 \int_{\mathbb{R}\setminus\Delta_\varepsilon} |x(t)|^2\, d\mu = \varepsilon^2 \int_{\mathbb{R}} |x(t)|^2\, d\mu == \varepsilon\|x\|^2.$$

This implies $\lambda \in \rho(A_\mu)$. Conversely, Let $\lambda \in \operatorname{supp}\mu$. Then for any $\varepsilon > 0$ we have $\mu(\Delta_\varepsilon) > 0$. Choose $\varepsilon_n \to 0$, $n \to \infty$ and introduce the sequence of functions x_n s.t. $x_n(t) = 1$ if $t \in \Delta_{\varepsilon_n}$ and $x_n(t) = 0$ if $t \notin \Delta_{\varepsilon_n}$. Then

$$\|A_\mu x - \lambda x\|^2 = \int_{\Delta_{\varepsilon n}} (t-\lambda)^2\, d\mu(t) \leq \varepsilon_n^2 \mu(\Delta_{\varepsilon_n} = \varepsilon_n^2\|x_n\|^2,$$

which implies $\lambda \in \sigma(A_\mu)$. □

Theorem 2. *The point spectrum $\sigma_p(A_\mu)$ coincides with the set of λ for which $\mu(\lambda) \neq 0$. Each of such λ is an eigenvalue.*

Proof. If $A_\mu x = \lambda x$ then $(t - \lambda x(t) = 0$ and thus $x(t) = 0$ μ-almost everywhere on $\mathbb{R}\backslash\{\lambda\}$. This implies that the (μ-)support of x is the one point set $\{\lambda\}$. If $\mu(\lambda) \not> 0$ then the function $x_\lambda(t) = 1$ if $t = \lambda$ and $x_\lambda(t) = 0$ if $t \neq \lambda$ is an eigenfunction corresponding to λ. □

Theorem 3. *The continuous spectrum of the operator A_μ coincides with non-isolated points of supp μ.*

Proof. Let λ be a non-isolated point. Such a point could be an eigenvalue. Assume that this is the case and let $F_\lambda = N(A_\mu - \lambda I)^\perp$ (if λ is not an eigenvalue then we assume that $F_\lambda = H$). Let x_λ is a function defined in the proof of Theorem 2 The condition $(y, x_\lambda) = 0$ is equivalent to

$$\int_{t=\lambda} y(t)\, d\mu = y(\lambda)\mu(\lambda) = 0$$

which implies that $y(\lambda) = 0$. Since λ is a non-isolated point of supp μ we can find a sequences $\lambda_n \to \lambda$ and $\varepsilon_n \to 0$ as $n \to \infty$ such that $\mu(\Delta_n) > 0$, where $\Delta_n = (\lambda_n - \varepsilon_n, \lambda_n + \varepsilon_n)$. Let now

$$y_n(t) = \begin{cases} 1, \text{ if } t \in \Delta_N, \\ 0, \text{ if } t \notin \Delta_N. \end{cases}$$

Then since $\lambda \notin \Delta_n$ we obtain that $y_n(\lambda) = 0$, so $y_n \in F_\lambda$. Now we obbtain

$$\begin{aligned} \|(A_\mu - \lambda I)y_n\|^2 = \int_{\Delta_n} (t-\lambda)^2\, d\mu(t) &\le (|\lambda - \lambda_n| + \varepsilon)^2 \mu(\Delta_n) \\ &= (|\lambda - \lambda_n| + \varepsilon)^2 \|y_n\|^2. \end{aligned}$$

Therefore λ is not a point from the resolvent set of the operator A_μ restricted on the set $D(A_\mu) \cap F_\lambda$. Since we exclude the eigenspace F_λ we arrive at $\lambda \in \sigma_c(A_\mu)$.

Suppose now that for some ε the interval $\Delta_\varepsilon = (\lambda - \varepsilon, \lambda + \varepsilon)$ does not belong to supp $\backslash\{\lambda\}$ which means $\mu(\Delta_\varepsilon) = \mu(\lambda)$. Then for $y \in D(A_\mu) \cap F_\mu$

we have (recall$y(\lambda) = 0$)

$$\|(A_\mu - \lambda I)y\|^2 = \int_{|t-\lambda|\geq\varepsilon} (t-\lambda)^2 |y(t)|^2 \, d\mu(t)$$
$$\geq \varepsilon^2 \int_{|t-\lambda|\geq\varepsilon} |y(t)|^2 \, d\mu(t) = \varepsilon^2 \|y\|^2.$$

This implies $\lambda \notin \sigma_c(A_\mu)$. □

Remark 1. A point $\lambda \in \mathbb{R}$ can belong to both $\sigma_p(A_\mu)$ and to $\sigma_c(A_\mu)$.

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2. Spectral Theorem

Let H be a Hilbert space and let $\mathcal{P}(H)$ be a set of orthogonal projections in H.

Definition. Let E be a mapping

$$E : \delta \to \mathcal{P},$$

defined on subsets δ from the sigma algebra of $\mathbb{R}$ and satisfying the following properties

- If $\delta_n \subset \mathbb{R}$ is a countable number of disjoint sets, $\delta = \cup_n \delta_n$, then $E(\delta) = s - \lim_{N\to\infty} \sum_{n=1}^{N} E(\delta_n)$
- $E(\mathbb{R}) = H$.

Then E is called a spectral measure.

Proposition 2. • $E(\delta_1)E(\delta_2) = E(\delta_1 \cap \delta_2) = E(\delta_2)E(delta_1)$.
- *If* $\delta_1 \subset \delta_2$ *then* $E(\delta_1) \leq E(\delta_2)$.

Let $\chi_{\delta_k}(t)$ be a characteristic function of the set δ_k, $\chi_{\delta_k}(t) = 1$, if $t \in \delta_k$ and $\chi_{\delta_k}(t) = 0$, if $t \notin \delta_k$. Let φ be a "step" function defined as

$$\varphi(t) = \sum_{|k|\leq n} c_k \chi_{\delta_k}(t), \quad c_k \in \mathbb{C}.$$

Then we can define the operator

$$\int_{\mathbb{R}} \varphi(\lambda) \, dE(\lambda) := \sum_{|k|\leq n} c_k E(\delta_k).$$

Theorem 4. *Let* A *be a self-adjoint operator in a Hilbert space* H. *Then there exists a unique spectral measure* E_A *such that*

$$A = \int_{\mathbb{R}} t \, dE(t) \tag{1}$$

which is called the spectral decomposition of the operator A.

Definition.

$$\operatorname{supp} E_A = \{\lambda : \forall \varepsilon > 0, E_A(\lambda - \varepsilon, \lambda + \varepsilon) \neq 0\}$$

Theorem 5. *Let* E_A *be the spectral measure of a self-adjoint operator* A. *Then*

- $\sigma(A) = \operatorname{supp} E_A$
- $\sigma_p(A) = \{\lambda \in \mathbb{R} : E_A(\lambda) \neq 0\}$ *and the eigenspace corresponding to* $\lambda \in \sigma_p(A)$ *is equal to* $E_A(\lambda)H$.
- $\sigma_c(A)$ *is the set of non-isolated points of* $\operatorname{supp} E_A$.

3. Compact Operators