LECTURE 14
RESOLVENT SET AND SPECTRUM

Let $G_A = \{(x, Ax),\ x \in D(A)\} \subset H \oplus H$ be the graph of the operator A.

We define $W : H \oplus H \to H \oplus H$ such that

$$W(x.h) = (-x, h), \qquad x, h \in H.$$

Obviously $W^2 = I$.

Theorem 1. *If* $\overline{D(A)} = H$*, then* $(WG_A)^\perp = G_{A^*}$.

Proof. The fact that $(h, y) \in H \oplus H$ is orthogonal to WG_A implies $-(x, h) + (Ax, y) = 0$. Thus $(Ax, y) = (x, h)$ and by definition of the adjoint operator we have $y \in D(A^*)$ and $h = A^*y$. □

Theorem 2. *The operator* A^* *is always closed.*

Proof. Indeed, the orthogonal complement to a linear subspace is always closed. Therefore $G_{A^*} = (WG_A)^\perp$, which the graph of the operator A^*, is a closed set. □

Theorem 3. *If* $\overline{D(A)} = H$*. Then the condition* $\overline{D(A^*)} = H$ *is equivalent to* A *being closable. Moreover, in this case* A^{**} *exists and* $A^{**} = \overline{A}$.

Proof. By Theorem 1 and since $W^2 = I$ we obtain

$$WG_{A^*} = W\Big[WG_A\Big]^\perp = \Big[W^2 G_A\Big]^\perp = \overline{G_A}.$$

Therefore

$$G_{A^{**}} = \Big[WG_{A^*}\Big]^\perp = \overline{G_A} = G_{\overline{A}}.$$

□

Theorem 4. *The subspaces* $\overline{R(A)}$ *and* $N(A^*)$ *are orthogonal in* H *and*

$$H = \overline{R(A)} \oplus N(A^*).$$

Proof. The element $y \in N(A^*)$ if and only if $(Ax, y) = 0$ for all $x \in D(A)$. This is equivalent to $y \in R(A)$. □

1. Spectrum and resolvent of a closed operator

Let A be a closed operator in a Hilbert space H.

Definition. $d_A = \mathrm{def} A = \dim [R(A)]^{\perp}$ is called the defect of the operator A.

Remark 1. By using Theorem 4 we immediately obtain that $d_A = \dim N(A^*)$.

Theorem 5. *Let us assume that* A *is a closed operator such for some constant* $C > 0$ $\|Ax\| \geq C\|x\|$ *for all* $x \in D(A)$. *Let* B *be an operator in* H *such that* $D(A) \subset D(B)$ *and for any* $x \in D(A)$

$$\|Bx\| \leq a\|Ax\|, \quad \textit{where}, \quad a < 1.$$

Then

- $A + B$ *is closed on* $D(A)$
- $\|(A + B)x\| \geq (1 - a)C\|x\|$
- $d_{A+B} = d_A$.

Proof. The graph G_A is closed w.r.t. the norm $|x|_A = \|x\| + \|Ax\|$. Therefore by using the triangle inequality we have

$$(1 - a)|x|_A \leq |x|_{A+B} \leq (1 + a)|x|_A.$$

Therefore the norms $|\cdot|_A$ and $|\cdot|_{A+B}$ — and thus since G_A is closed then G_{A+B} is closed. This implies that $A + B$ is closed. Now

$$\|(A + B)x\| \geq \|Ax\| - \|Bx\| \geq (1 - a)\|Ax\| \geq (1 - a)C\|x\|,$$

which proves the second statement of the theorem Assume for a moment that $d_{A+B} < d_A$. Then there exists $f \in R(A)^{\perp}$, $f \neq 0$, such that $f \perp R(A + B)^{\perp}$. This implies $f \in R(A + B)$ and therefore there exists $y \in D(A)$ s.t. $f = (A + B)y$. Since $f \perp R(A)$ we have $(f, Ay) = 0$.

$$\|Ay\|^2 = (Ay, Ay) = -(By, Ay) \leq \|By\|\|Ay\| \leq a\|Ay\|^2,$$

which gives a contradiction.

If we assume that $d_A + B > d_A$, then we can find $f = Ay$, $f \neq$ s.t. $f \perp R(A + B)$ and thus $(f, (A + B)y) = 0$. Finally

$$\|Ay\|^2 = (Ay, Ay) = -(By, Ay) \leq \|By\|\|Ay\| \leq a\|Ay\|^2.$$

□

Corollary 1. *Let* A *be an operator in* H *satisfying the condition* $\|Ax\| \geq C\|x\|$, $x \in D(A)$ *and let* B *be a bounded operator s.t.* $\|B\| < C$. *Then*

$$d_{A+B} = d_A.$$

Proof.

$$\|Bx\| \leq \|B\|\|x\| \leq \|B\|C^{-1}\|Ax\|.$$

We now apply Theorem 5 with $a = \|B\|C^{-1}$. □

Definition.

The defect of the operator $A - \lambda I$ is denoted by $d_A(\lambda)$ and called the defect of A at λ.

If $A - \lambda I$ has a bounded inverse on its image $(A - \lambda I)(H)$ $\Big($namely $\|(A - \lambda I)x\| \geq C\|x\|$ for some $C > 0\Big)$ then λ is called a quasi-regular point of A.

All such points are denoted by $\hat{\rho}(A)$.

Lemma 1. *Let* A *be a closed operator in* H *such that* $(\|A - \lambda I)x\| \geq C_0\|x\|$ *for some* $C_0 > 0$ *and* $\forall x \in D(A)$. *Then*

$$\mathbb{D} := \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < C_0\} \subset \hat{\rho}(A)$$

and $d_A(\lambda)$ *is constant on* $\mathbb{D}$.

Proof. If we write

$$A - \lambda I = (A - \lambda_0) + (\lambda_0 - \lambda)I$$

then we complete the proof by using Corollary 1. □

We now immediately obtain the following result:

Theorem 6. *The set* $\hat{\rho}(A) \subset \mathbb{C}$ *is open and the value of* d_A *is constant on each connected component of* $\hat{\rho}(A)$.

Definition. If $d_A(\lambda) = 0$ for some $\lambda \in \hat{\rho}(A)$, then λ is called a regular point of A. In this case the operator $(A - \lambda I)-1$ is bounded.

The set of all regular points of A is called a resolvent set and denoted by $\rho(A)$.

Remark 2. The set $\rho(A)$ is open.

Definitions.

- The set $\sigma(A) = \mathbb{C} \setminus \rho(A)$ is called the spectrum of theoperator A.
- $\hat{\sigma}(A) = \mathbb{C} \setminus \hat{\rho}(A)$ is called the core of the spectrum.
- The set $\sigma_p(A) = \{\lambda \in \mathbb{C} : N(A - \lambda I) \neq \{0\}\}$ is called the point spectrum of A and $\lambda \in \sigma_p$ is called the eigenvalue of A.
- The set $\sigma_c(A) = \{\lambda \in \mathbb{C} : R(A - \lambda I) \neq \overline{R(A - \lambda I)}\}$ is called the continuous spectrum of A.

Example.

1. Let $A = \frac{1}{i}\frac{d}{dt}$ defined on $D(A) = \{x : \int_{-1}^{1}(|x'(t)|^2 + |x(t)|^2)\,dt, \} \subset L^2(-1, 1)$. $D(A)$ is a dense in $L^2(-1, 1)$ set. Solutions of the equation $Ax = \lambda x$ are $x(t) = e^{ikt}$ and $\sigma_p(A) = k$, $k = 0, \pm 1, \pm 2, \ldots$.

2. Let A be defined as $Ax(t) = tx(t)$ in $L^2(0, \infty)$. The operator $A$is bounded and its spectrum is continuous and equal $\sigma_c(A) = [0, \infty)$. (For the proof see Lecture 15).

Definition. Let $\overline{D(A)} = H$. An operator A is called symmetric if $(Ax, y) = (x, Ay)\ \forall x, y \in D(A)$.

It follows from the definition that $A \subset A^*$. Therefore A can be closed and in particular $\overline{A} = A^{**}$.

Home exercises.

1. Let $A_0 = \frac{1}{i}\frac{d}{dt}$ defined the class of functions $\{x : x \in C_0^\infty(\mathbb{R})\}$. Show that A_0 is symmetric and closable.

2. Let $A = \frac{1}{i}\frac{d}{dt}$ defined on $\{x : \int_{-1}^{1}(|x'(t)|^2 + |x(t)|^2)\,dt, \} \subset L^2(-1, 1)$ such that $x(-1) = x(1)$. Show that A is self-adjoint.

3. Describe the closure of the class of functions $\{x : x \in C_0^\infty(\mathbb{R})\}$ with respect to

$$\int_{\mathbb{R}} |x'(t)|^2\,dt.$$