LECTURE 13
POSITIVE OPERATORS & UNBOUNDED OPERATORS

Theorem 1. *Let* A_n *be a bounded monotone sequence of operators. Then* $\{A_n\}$ *is strongly convergent.*

Proof. Assume, for example, that $A_n \searrow$. Since $\sup_n \|A_n\| < \infty$ we obtain that for any $x \in H$ the sequence $(A_n x, x)$ is convergent. Therefore due to "polarisation"

$$(A_n x, y) = \frac{1}{4}\Big[\Big(A_n(x+y), x+y\Big) - (A_n(x-y), x-y\Big)\Big]$$

we have that $\lim_n (A_n x, y)$ exists for any $x, y \in H$. Such a limit defines a self-adjoint bounded operator A in H. Denote by $c = \sup_n \|A_n - A\|$. Then by using Lemma 1, Lecture 12, we find

$$\|A_n x - Ax\| \leq c[(A_n x, x) - (Ax, x)] \to 0 \qquad \forall x \in H.$$

□

Definition. A square root of a positive operator A is a self-adjoint operator B such that $B^2 = A$.

Theorem 2 (Av.Fr. 6.6.4)**.** *Every positive operator* A *has a unique positive square root* B *such that* B *commutes with any linear operator that commutes with* A.

Proof. See Av.Fr. page 223. □

Definition. Let A be a bounded operator. We define $|A| = \sqrt{A^* A}$.

Remark 1.
- $|\lambda A| = |\lambda||A|$ if $\lambda \in \mathbb{C}$.
- In general $|AB| = |A||B|$ or $|A| = |A^*|$ is false.
- In general $|A + B| \leq |A| + |B|$ is not true.

Definition. A bounded operator U in H is called unitary if $UH = H$ and $U^* U = I$, where I is the identity, $Ix = x$.

Example. Let A be a shift operator in l^2 such that $A(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots)$. Then A^* could be defined by $A^*(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$. Clearly $|A| = \sqrt{A^*A} = I$. If the formula $A = U|A|$ would hold true for some unitary operator U, then U should be equal to A. However, A is not unitary since $(1, 0, 0, \dots) \notin H$.

Definition. A bounded operator U is called isometry if $\|Ux\| = \|x\|$ for all $x \in H$. U is called a partial isometry if U is an isometry when restricted to $(\operatorname{Ker} U)^{\perp}$.

Theorem 3 (Polar decomposition)**.** *Let A be a bounded operator in H. There exists a partial isometry U such that $A = U|A|$. The operator U is uniquely determined by the condition* $\operatorname{Ker} U = \operatorname{Ker} A$.

Proof. Let $U : \operatorname{Ran}|A| \to \operatorname{Ran} A$ such that $U(|A|x) = Ax$. Then

$$\||A|x\|^2 = (x, |A|^2 x) = (x, A^*Ax) = \|Ax\|^2.$$

Clearly $|A|x = 0$ is equivalent to $Ax = 0$ and therefore $\operatorname{Ker}|A| = \operatorname{Ker} A$ and thus $\operatorname{Ker} U = \operatorname{Ker} A$. □

1. Unbounded operators

Let $A : H \to H$ be an operator in H with domain $D(A)$ and range $R(A) := A(H)$. By $N(A)$ we denote the kernel of the operator A, $N(A) = \{x \in D(A) : Ax = 0\}$.

The operator A is invertible if and only if $N(A) = 0$. In this case $D(A^{-1}) = R(A)$ and $R(A^{-1}) = D(A)$.

The sesqui-linear form

$$(x, y)_A = (x, y) + (Ax, Ay), \qquad x, y \in D(A),$$

define pre-Hilbert space (a space with a scalar product which is not necessary complete).

Theorem 4. *The operator A has a bounded inverse if and only if there is a constant $c > 0$ such that $\|Ax\| > c\|x\|$. The biggest possible constant c in the latter inequality is equal to $\|A^{-1}\|^{-1}$.*

Proof. Let us point out that $\|Ax\| > c\|x\|$ implies $N(A) = 0$ and therefore A^{-1} exists. Let $y = Ax$. Then $\|y\| \geq c\|A^{-1}\|y$ and thus $c^{-1}\|y\| \geq \|A^{-1}\|\|y\|$, or $\|A^{-1}\| \leq c^{-1}$.

If A^{-1} exists and bounded then $\|A^{-1}y\| \leq c^{-1}\|Ay\|$, where $c = \|A^{-1}\|^{-1}$. □

Definition. We say that two unbounded operators A_1 and A_2 are equal $A_1 = A_2$ if $D(A_1) = D(A_2)$ and $A_1x = A_2x$ for any $x \in D(A_1) = D(A_2)$.

If $D(A_1) \subset D(A_2)$ and $A_1x = A_2x$ for any $x \in D(A_1)$, then A_2 is called an extension of A_1.

Let $G_A = \{(x, Ax) : x \in D(A)\} \subset H \oplus H$. The scalar product

$$\Big((x, Ax), (y, Ay)\Big) := (x, y) + (Ax, Ay)$$

define a pre-Hilbert structure in G_A.

Theorem 5. *An operator A is closed iff G_A is closed in H and iff for any sequence $x_n \in D(A)$*

$$\lim_n x_n = x, \quad \lim_n Ax_n = y \qquad \Longrightarrow \qquad x \in D(A) \quad \& \quad y = Ax.$$

Proof. See Th. 4.7.1 Av.Fr. □

2. Adjoint operators

Definition. Let A be a linear operator in H and let $\overline{D(A)} = H$. An element $y \in H$ is said to be in $D(A^*)$ if there exists h such that

$$(Ax, y) = (x, h), \qquad \forall x \in D(A).$$

In this case we define the adjoint operator A^* as $A^*y = h$ and thus $(Ax, y) = (xA^*y)$.

If $\overline{D(A)} \neq H$ then the element h is not unique and we are not able to define A^*.

Home exercises.

1. Show that an operator U is a partial isometry if and only if the operators $P_1 = U^*U$ and $P_2 = UU^*$ are projections.

2. (Av.Fr. &.6.1) Find the positive square root of the operator $Ax(t) = a(t)x(t)$ in $L^2(0, 1)$ assuming that $a(t) \geq 0$.

3. Prove that if $A \geq B \geq 0$, then $\sqrt{A} \geq \sqrt{B}$ (Heinz's inequality).
(Hint: $(A + tI)^{-1} \geq (B + tI)^{-1}$ for any $t > 0$).