LECTURE 12
COMPACT & BOUNDED OPERATORS IN HILBERT SPACE

1. Finite rank operators

Definition. T is said to be of rank r ($r < \infty$) if $\dim T(H) = r$. The class of operators of rank r is denoted by K_r and $K := \cup_r K_r$.

Theorem 1. $T \in K_r$ *iff* $T^* \in K_r$.

Proof. Let $T \in K_r$ and let $u_1, u_2, \dots, u_r$ be an orthonormal basis in $T(H)$. Then for any $x \in H$ we have

$$Tx = \sum_{k=1}^{r} (Tx, u_k)u_k = \sum_{k=1}^{r} (x, T^*u_k)u_k.$$

Denote $v_k = T^*u_k$, then $T = \sum_{k=1}^{r} (\cdot, v_k)u_k$. Moreover

$$(Tx, y) = \sum_{k=1}^{r} ((x, v_k)u_k, y) = \sum_{k=1}^{r} (x, (y, u_k)v_k) = (x, T^*y).$$

Therefore $T = \sum_{k=1}^{r} (\cdot, u_k)v_k$ and thus $T^* \in K_r$. □

Theorem 2. *The uniform closure of the class of finite rank operators* K *coincides with* S_∞.

Proof. Let $T \in S_\infty$. Then T maps the set $B = \{x : \|x\| \le 1\}$ onto a relatively compact set. For any $\varepsilon > 0$ there exists a finite set of elements $\{y_k\}_{k=1}^{r}$ such that for any $y \in T(B)$ we have $\min \|y - y_k\| \le \varepsilon$. Let P be the projection on the subspace spanned by y_k. Clearly rank $P \le r$. Thus for any x s.t. $\|x\| \le 1$ we obtain

$$\|Tx - PTx\| \le \min_k \|Tx - y_k\| \le \varepsilon.$$

□

Remark 1. Uniform closure cannot be replaced by the strong closure.

Theorem 3. *The strong closure of* $K(H)$ *coincides with the class of all bounded operators.*

Proof. Let $\{u_k\}_{k=1}^{\infty}$ be an orthonormal basis in H and let P_n be the projectors on the subspaces spanned by $\{u_k\}_{k=1}^{n}$. Then for any $x \in H$, $\|P_n x - x\| \to 0$ which means that s-lim $P_n = I$. Thus s-lim $P_n T = T$ for any bounded operator T. □

2. INTEGRAL OPERATORS

Theorem 4. *Let* $K : L^2(\Omega) \to L^2(\Omega)$, $\Omega \in \mathbb{R}$, *be an integral operator*

$$Kf(x) = \int_\Omega K(x,y) f(y)\, dy,$$

such that

$$\int_\Omega \int_\Omega |K(x,y)|^2\, dx dy < \infty.$$

Then K *is compact.*

Proof. Let $\{u_j\}_{j=1}^{\infty}$ be an orthonormal basis in $L^2(\Omega)$. Then

$$K(x,y) = \sum_{j=1}^{\infty} K_j(y) u_j(x), \quad \text{where} \quad K_j(y) = \int_\Omega K(x,y)\overline{u_j(x)}\, dx$$

for almost all y. Due to the Parseval identity we have for almost all y

$$\int_\Omega |K(x,y)|^2\, dx = \sum_{j=1}^{\infty} |K_j(y)|^2$$

and

$$\int_\Omega \int_\Omega |K(x,y)|^2\, dx dy = \sum_{j=1}^{\infty} \int_\Omega |K_j(y)|^2\, dy. \tag{1}$$

We now define the following operator of rank N

$$K_N f(x) = \int_\Omega K_N(x,y) f(y)\, dy,$$

where $K_N(x,y) = \sum_{j=1}^{N} K_j(y) u_j(x)$. By Cauchy-Schwartz inequality we obtain

$$\begin{aligned}\|(K - K_N) f\|^2 &= \int_\Omega \left| \int_\Omega \Big(K(x,y) - K_N(x,y) \Big) f(y)\, dy \right|^2 dx \\ &\le \int_\Omega \int_\Omega |K(x,y) - K_N(x,y)|^2\, dx dy\, \|f\|^2\end{aligned}$$

Thus by using that the right hand side in (1) is absolutely convergent, we find

$$\begin{aligned}
\|(K-K_N)\|^2 &\le \int_\Omega\int_\Omega |K(x,y)-K_N(x,y)|^2\,dxdy \\
&= \int_\Omega\int_\Omega |K(x,y)|^2\,dxdy - \int_\Omega\int_\Omega K(x,y)\sum_{j=1}^N \overline{K_j(y)u_j(x)}\,dxdy \\
&\quad - \int_\Omega\int_\Omega \overline{K(x,y)}\sum_{j=1}^N K_j(y)u_j(x)\,dxdy + \sum_{j=1}^N\int_\Omega |K_j(y)|^2\,dy \\
&= \int_\Omega\int_\Omega |K(x,y)|^2\,dxdy - \sum_{j=1}^N\int_\Omega |K_j(y)|^2\,dy \to 0, \quad \text{as} \quad N\to\infty.
\end{aligned}$$

□

3. Bounded Self-adjoint Operators

Definition. A bounded operator $T : H \to H$ is said to be self-adjoint if $\forall x, y \in H$

$$(Tx, y) = (x, Ty), \qquad (A = A^*).$$

Theorem 5 (Av.Fr. 6.5.1)**.** *Let* T : *be a bounded self-adjoint operator in a Hilbert space* H. *Then*

$$\|T\| = \sup_{\|x\|=1} |(Tx, x)|.$$

Proof. Clearly if $\|x\| = 1$, then

$$|(Tx, x)| \le \|Tx\|\|x\| = \|Tx\| \le \|T\|$$

and therefore $\sup_{\|x\|=1} |(Tx, x)| \le \|T\|$.

In order to proof the inverse inequality we consider $z \in H$, $\|z\| = 1$, $Tz \ne 0$ and $u = Tz/\lambda$, where $\lambda = \|Tz\|^{1/2}$. If we denote by $\alpha := \sup_{\|x\|=1} |(Tx, x)|$, then

$$\begin{aligned}
\|Tz\|^2 &= \Big(T(\lambda z), u\Big) = \frac{1}{4}\Big[\Big(T(\lambda z+u), \lambda z+u\Big) - \Big(T(\lambda z-u), \lambda z-u\Big)\Big] \\
&\le \frac{\alpha}{4}\Big[\|\lambda z+u\|^2 + \|\lambda z-u\|^2\Big] = \frac{\alpha}{2}\Big[\|\lambda z\|^2 + \|u\|^2\Big] \\
&\qquad \frac{\alpha}{2}\Big[\|\lambda\|^2 + \|Tz\|\Big] = \alpha\|Tz\|.
\end{aligned}$$

This implies that for any $z \in H$, $\|z\| = 1$ we have $\|Tz\| \leq \alpha$ and hence $\|T\| \leq \alpha = \sup_{\|x\|=1} |(Tx, x)|$. □

Definition. If $Tx = \lambda x$, $x \in H$, $x \neq 0$, then x is called an eigenvector and λ is called an eigenvalue for the operator T.

Theorem 6 (Av.Fr. 6.5.2). *Let* $\{\lambda_j\}$ *and* $\{x_j\}$ *be eigenvalues and eigenvectors for a self-adjoint operator* T *in* H. *Then*

- *All* λ_j *are real.*
- *If* $\lambda_j \neq \lambda_k$ *then the corresponding eigenvectors* x_j *and* x_k *are orthogonal.*

Proof. Let $Tx = \lambda x$. Then

$$\lambda \|x\|^2 = (Tx, x) = (x, Tx) = \overline{(Tx, x)} = \overline{\lambda} \|x\|^2.$$

Let now $Ty = \mu y$, $\lambda \neq \mu$. Then since T is self-adjoint we have

$$0 = (Tx, y) - (x, Ty) = \lambda(x, y) - \mu(x, y) = (\lambda - \mu)(x, y).$$

□

Theorem 7. *Let* T_1 *and* T_2 *be two bounded self-adjoint operators. The product* $T := T_1 T_2$ *is self-adjoint iff the operators* T_1 *and* T_2 *commute* $(T_1 T_2 = T_2 T_1)$.

Proof.

$$T^* = T_2^* T_1^* = T_2 T_1 = T_1 T_2 = T.$$

□

Definition.

- A bounded operator T in H is called positive ($T \geq 0$) if for any $x \in H$, $(Tx, x) \geq 0$.
- A bounded operator T in H is called strictly positive ($T > 0$) if for any $x \in H$, $(Tx, x) > 0$.
- A bounded operator T in H is called positive definite if there exist a positive constant $m > 0$ such that

$$(Tx, x) \geq m \|x\|^2, \qquad \forall x \in H.$$

If T is positive then we can define an inner product $< x, y >= (Tx, y)$. The the Schwartz inequality gives us

(2) $$(Tx, y)^2 =< x, y >^2 \leq < x, x >< y, y >= (Tx, x)(Ty, y).$$

Lemma 1. *If* $T \geq 0$ *then* $\|Tx\|^2 \leq \|T\|(Tx, x)$.

Proof.

$$(Tx, y)^2 \leq (Tx, x)(Ty, y) \leq (Tx, x)\|T\|\|y\|^2.$$

The proof is complete if we substitute into the latter inequality $y = Tx$. □

Definition. We say that $B \geq A$ if $B - A \geq 0$.

For any bounded operator T there exist two unique self-adjoint operators A and B such that $T = A + iB$. Then $T^* = A - iB$ and the operators A and B could be found via

$$A = \frac{T + T^*}{2}, \qquad B = \frac{T - T^*}{2i}.$$

It is clearly that $T^*T \geq 0$ and $TT^* \geq 0$. The operators T and T^* do not always commute.

Definition. If $T^*T = TT^*$ then T is called normal.

Home exercises.

1. Let $l^2 = \left\{x = \{x_j\}_{j=1}^{\infty}, x_j \in \mathbb{C} : \sum_j |x|^2 < \infty\right\}$. Define $T : l^2 \to l^2$ such that

$$T(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots).$$

Show that $T^*T \neq TT^*$.

2. Let P_L and P_M be two projectors on the the subspaces $L, M \subset H$. Show that $P_L \leq P_M$ *iff* $L \subset M$.