LECTURE 11
COMPACT OPERATORS IN HILBERT SPACE

1. Weak convergency

Definition.
Let H be a Hilbert space, $x_n, x \in H$.
We say that $x_n \to x$ weakly $\left[x = w\text{-}\lim_{n\to\infty} x_n\right]$ if for any $h \in H$ we have $(x_n, h) \to (x, h)$.

Remark 1. If $\|x_n - x\| \to 0$ then $x = w\text{-}\lim_{n\to\infty} x_n$.

Example. Let $\{u_n\}$ be an orthonormal system. Then $w - \lim_{n\to\infty} u_n = 0$. Indeed, for any $x \in H$, $x_n = (x, u_n)$ are its Fourier coefficients converging to zero.

Lemma 1. *Let* $\{x_n\}$ *be a weakly convergent sequence. Then* $\{x_n\}$ *is bounded.*

Proof. For any $y \in H$ we have $(x_n, y) \to (x, y)$, $x \in H$. Therefore the sequence $\{(x_n, y)\}$ is bounded. Now the lemma follows from the principle of uniform boundedness Th. 4.5.1 (AvFr). □

Lemma 2. *If* $y_n \to y$ *and* $w\text{-}\lim_{n\to\infty} x_n = x$*, then* $(x_n, y_n) \to (x, y)$.

Proof.

$$|(x_n, y_n) - (x, y)| \leq \|x_n\|\|y_n - y\| + |(x_n, y) - (x, y)| \to 0.$$

□

2. Sesqui-linear forms

Definition. A functional $\Phi : H \times H \to \mathbb{C}$ is called sesqui-linear form if $\Phi(x, y)$ it is linear w.r.t. x, anti-linear w.r.t. y and

$$\|\Phi\| := \sup_{\|x\|=\|y\|=1} |\Phi(x, y)| < \infty.$$

Remark 2. Clearly $|\Phi(x, y)| \leq \|\Phi\|\|x\|\|y\|$.

Examples. Let T and S be bounded operators in H. Then $\Phi(x,y) = (Tx,y)$ and $\Phi(x,y) = (x,Sy)$ are sesqui-linear forms.

Definition. $\Phi(x) = \Phi(x,x)$ is called a quadratic form.

Remark 3. By using the same argument as in Th. 6.1.5.(Av.Fr) we find

$$(1) \qquad 4\Phi(x,y) = \Phi(x+y) - \Phi(x-y) + i\Phi(x+iy) - i\Phi(x-iy).$$

Definition. A form Φ is called Hermitian if

$$\Phi(y,x) = \overline{\Phi(x,y)}.$$

Theorem 1. *The form $\Phi(x,y)$ is Hermitian iff the quadratic form $\Phi(x)$ is real.*

Proof. $\Phi(y,x) = \overline{\Phi(x,y)}$ implies $\Phi(x,x) = \overline{\Phi(x,x)}$. If $\Phi(x)$ is real then by using (1) we obtain that $\Phi(x,y)$ is Hermitian. □

Theorem 2. *If $\Phi(x,y) = (Tx,y) = (x,Sy)$, then $\|\Phi\| = \|T\| = \|S\|$.*

Proof. Since $|\Phi(x,y)| \leq \|\Phi\|\|x\|\|y\|$ the functional $\Phi(\cdot,y)$ is continuous. Then by Riesz theorem (Th. 6.2.4) there exists $h \in H$ s.t. $\Phi(x,y) = (x,h)$, $\forall x \in H$. Define now $S:\ y \to h$. The operator S is linear. Indeed

$$\begin{aligned}(\cdot, S(\alpha_1 y_1 + \alpha_2 y_2)) &= \Phi(\cdot, \alpha_1 y_1 + \alpha_2 y_2) \\ &= \bar{\alpha}_1 \Phi(\cdot, y_1) + \bar{\alpha}_2 \Phi(\cdot, y_2) = \bar{\alpha}_1(\cdot, Sy_1) + \bar{\alpha}_2(\cdot, Sy_2) \\ &= (\cdot, \alpha_1 Sy_1 + \alpha_2 Sy_2).\end{aligned}$$

This implies $\|Sy\| = \|h\| \leq \|\Phi\|\|y\|$ and thus $\|S\| \leq \|\Phi\|$.

On the other hand $|\Phi(x,y)| \leq \|x\|\|Sy\| \leq \|S\|\|x\|\|y\|$ which gives us $\|\Phi\| \leq \|S\|$. □

Corollary 1. *For any bounded operator T in a Hilbert space H $\|T\| = \|T^*\|$.*

Theorem 3. *A linear bounded operator T in H is defined by its quadratic form (Tx,x).*

Proof. Suppose that $(T_1x,x) = (T_2x,x)$. Then by using (1) we find that $(T_1x,y) = (T_2x,y)$, $\forall x,y \in H$. Hence $T_1 = T_2$. □

Definition. Let $\{T_n\}$ be a sequence of bonded operators in H

- $T_n \to T$ uniformly if $\|T_n - T\| \to 0$, as $n \to \infty$.
- $T_n \to T$ strongly if $\|T_n x - Tx\| \to 0$, as $n \to \infty$, $\forall x \in H$.
- $T_n \to T$ weakly if $(T_n x, y) \to (Tx, y)$, as $n \to \infty$, $\forall x, y \in H$.

Theorem 4. *If* $w\text{-}\lim T_n = T$, *then* $w\text{-}\lim T_n^* = T^*$.

Proof.

$$\big((T_n - T)x, y\big) \to 0 \quad \Longrightarrow \quad \big((T_n^* - T^*)x, y\big) = \big(x, (T_n - T)y\big) = \overline{\big((T_n - T)y, x\big)} \to 0.$$

□

Remark 4. $s\text{-}\lim T_n = T$ does not imply $s\text{-}\lim T_n^* = T^*$.
Indeed, let $h \in H$, $\|h\| = 1$ and let $\{u_n\}$ be an orthonormal system in H. Define $T = (\cdot, u_n)h$. Then

$$(T_n x, y) = ((x, u_n)h, y) = (x, (y, h)u_n) \quad \Longrightarrow \quad T^* = (\cdot, h)u_n.$$

Since $\|T_n x\| = |(x, u_n)| \to 0$ we have $s\text{-}\lim_{n\to\infty} T_n = 0$. However, $\|T_n^* h\| = \|u_n\| = 1 \not\to 0$.

3. Compact Operators

Definition. A bounded operator $T : H \to H$ is called compact if it maps bounded sets onto relatively compact. We shall denote the class of compact operators by S_∞.

Theorem 5 (AFr Th. 5.1.1.). *If* $T \in S_\infty$ *then it maps weakly convergent sequences into convergent sequences.*

Proof. see the proof from AFr Th. 5.1.1 given for more general spaces. □

Theorem 6 (AFr Th. 5.1.2.). *Let* $T_n : H \to H$ *be sequence of compact operators uniformly convergent to* T. *Then* T *is also compact.*

Theorem 7. $T \in S_\infty$ *iff* $T^*T \in S_\infty$.

Proof. If $T \in S_\infty$ then T^* is bounded and therefore $T^*T \in S_\infty$. If $T^*T \in S_\infty$ and $w\text{-}\lim x_n = 0$, then $s\text{-}\lim T^*Tx_n = 0$. Then by Lemma 2 we obtain $(x_n, T^*Tx_n) = \|Tx_n\|^2 \to 0$. □

Theorem 8. $T|inS_\infty$ *iff* $T^* \in S_\infty$.

Proof. $T^* \in S_\infty$ implies $T^*T \in S_\infty$ and thus $T \in S_\infty$. □

4. Finite rank operators

Definition. T is said to be of rank r ($r < \infty$) if $\dim T(H) = r$. The class of operators of rank r is denoted by K_r and $K := \cup_r K_r$.

Theorem 9. $T \in K_r$ *iff* $T^* \in K_r$.

Proof. Let $T \in K_r$ and let $u_1, u_2, \dots, u_r$ be an orthonormal basis in $T(H)$. Then for any $x \in H$ we have

$$Tx = \sum_{k=1}^{r} (Tx, u_k)u_k = \sum_{k=1}^{r} (x, T^* u_k)u_k.$$

Denote $v_k = T^* u_k$, then $T = \sum_{k=1}^{r} (\cdot, v_k)u_k$. Moreover

$$(Tx, y) = \sum_{k=1}^{r} ((x, v_k)u_k, y) = \sum_{k=1}^{r} (x, (y, u_k)v_k) = (x, T^* y).$$

Therefore $T = \sum_{k=1}^{r} (\cdot, u_k)v_k$ and thus $T^* \in K_r$. □

Theorem 10. *The uniform closure of the class of finite rank operators* K *coincides with* S_∞.

Proof. Let $T \in S_\infty$. Then T maps the set $B = \{x : \|x\| \le 1\}$ onto a relatively compact set. For any $\varepsilon > 0$ there exists a finite set of elements $\{y_k\}_{k=1}^{r}$ such that for any $y \in T(B)$ we have $\min \|y - y_k\| \le \varepsilon$. Let P be the projection on the subspace spanned by y_k. Clearly $\operatorname{rank} P \le r$. Thus for any x s.t. $\|x\| \le 1$ we obtain

$$\|Tx - PTx\| \le \min \|Tx - y_k\| \le \varepsilon.$$

□

Remark 5. Uniform closure cannot be replaced by the strong closure.

Theorem 11. *The strong closure of* $K(H)$ *coincides with the class of all bounded operators.*

Proof. Let $\{u_k\}_{k=1}^{\infty}$ be an orthonormal basis in H and let P_n be the projectors on the subspaces spanned by $\{u_k\}_{k=1}^{n}$. Then for any $x \in H$, $\|P_n x - x\| \to 0$ which means that $\text{s-lim}\, P_n = I$. Thus $\text{s-lim}\, P_n T = T$ for any bounded operator T. □

Home exercises.

1. Let $T : L^2(0,1) \to L^2(0,1)$ be an operator defined by

$$Tu(x) = \int_0^1 K(x,y)u(y)\,dy,$$

where $K \in L^2((0,1)\times(0,1))$. Show that T is compact.

2. Let $Tx(t) = tx(t)$, $0 \le t \le 1$, $T : L^2(0,1) \to L^2(0,1)$. Is T compact?

3. Show that the integral operator $T : L^2(0,\infty) \to L^2(0,\infty)$ defined by

$$Tu(x) = \int_0^\infty (x+y)^{-1}u(y)\,dy$$

is bounded b ut not compact. Find its norm.