LECTURE 10
CH. 6.4 FROM A. FRIEDMAN

Definition.
Let H be a Hilbert space. A set $K \subset H$ is called orthonormal if for each element of $x \in K$ we have $\|x\| = 1$ and and any two elements from K are orthogonal. An orthonormal set K is called complete if there exists no nonzero elements that are orthogonal to K.

Theorem. (AFr 6.4.1.) (Bessel's inequality)
Let $\{x_n\}_{n=1}^{\infty}$ be an orthonormal set in a Hilbert space H. Then for any $x \in H$

$$\sum_{n=1}^{\infty} |(x, x_n)|^2 \leq \|x\|^2.$$

Theorem. (AFr 6.4.2.)
Let $\{x_n\}_{n=1}^{\infty}$ be an orthonormal set in a Hilbert space H and let $\{\lambda_n\}$ be any sequence of scalars. Then for any m

$$\left\| x - \sum_{n=1}^{\infty} \lambda_n x_n \right\| \geq \left\| x - \sum_{n=1}^{\infty} (x, x_n) x_n \right\|.$$

Definition.
A set K is called an orthonormal basis of H if $K$is an orthonormal set and if for any $x \in H$

$$x = \sum_{y \in K} (x, y) y.$$

Definition.
A space X is called separable if it contains a countable dense set.

Lemma. (AFr 6.4.7.)
Any orthonormal basis in a separable Hilbert space is countable.

Theorem.
The set $\{x_n\}$ is an orthonormal basis in a separable Hilbert space H *iff*

$$\|x\|^2 = \sum_{n=1}^{\infty} |(x, x_n)|^2$$

for any $x \in H$.

Lemma. (AFr 6.4.8.)
Any two infinite dimensional separable Hilbert spaces are isometrical isomorphic.

Corollary. (AFr 6.4.9.)
Any infinite dimensional separable Hilbert space is isometrical isomorphic to $L^2(0,1)$.

Examples.

1. $H = L^2(-\pi,\pi)$ with an orthonormal basis $u_n(t) = e^{int}/\sqrt{2\pi}$, $n \in \mathbb{Z}$.

2. $H = H^1(-\pi,\pi)$ - Sobolev space with the scalar product:

$$(u,v) = \int_{-\pi}^{\pi} \Big(u'(t)\overline{v'(t)} + u(t)\overline{v(t)}\Big)\,dt.$$

The set of functions $u_n(t) = e^{int}/\sqrt{2\pi}$ is orthogonal set but not normal in $H^1(-\pi,\pi)$.
Question: Is the set $\{u_n\}$ basis in the Hilbert space $H^1(-\pi,\pi)$?
Namely, is there a function $h(t)$ such that h is orthogonal to all u_n?
Assume that there exists such a function h. Then

$$\begin{aligned}(1)\quad 0 &= \int_{-\pi}^{\pi} \Big(h'(t)(-ine^{-int} + h(t)e^{-int}\Big)\,dt\\ &= (1+n^2)\int_{-\pi}^{\pi} h(t)e^{-int}\,dt - inh(t)e^{-int}\Big|_{-\pi}^{\pi}\\ &= (1+n^2)\,c_n - in(-1)^n\big(h(\pi) - h(-\pi)\big),\end{aligned}$$

where

$$c_n = \int_{-\pi}^{\pi} h(t)e^{-int}\,dt.$$

From (1) we obtain

$$(2)\qquad c_n = \frac{i}{\sqrt{2\pi}}\,\frac{n}{1+n^2}\,(-1)^n\Big(h(\pi) - h(-\pi)\Big), \qquad n = \mathbb{Z}.$$

We not define

$$(3)\qquad h(t) = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty} c_n e^{int}.$$

3. $H = L^(-1,1)$ with an orthonormal basis

$$u_n(t) = c_k\frac{d^k}{dt^k}(t^2-1)^k, \qquad \text{where } c_k = \frac{\sqrt{k+1/2}}{2^k k!}, \quad k = 0,1,\ldots.$$

4. $H = L^2(\mathbb{R})$

$$u_k(t) = (-1)^k c_k e^{t^2/2} \frac{d^k}{dt^k} e^{-t^2}, \qquad \text{where} \quad c_k = (2^k k!)^{-1/2} \pi^{-1/4}, \quad k = 0, 1, 2, \ldots.$$

5. Let $\mathbb{D}$ be a unit ball in the complex plane

$$\mathbb{D} = \{z = x + iy : |z| < 1\}.$$

Let $A^2(\mathbb{D})$ be the set of $L^2(\mathbb{D})$ analytic functions in $\mathbb{D}$. The space A^2 is a Hillbert space with an orthonormal basis

$$u_k(z) = \pi^{-1/2}(k+1)^{1/2} z^k, \qquad k = 0, \pm 1, \pm 2, \ldots.$$

Indeed, let $z = re^{it}$. Then

$$(u_k, u_l) = \pi^{-1}(k+1)^{1/2}(l+1)^{1/2} \int_0^{2\pi} \int_0^1 e^{i(k-l)t} r^{k+l+1} dr dt = \delta_{kl}.$$

6. $H = F^2$ - the Fock space, the space of entire function s.t.

$$\|f\|^2 = \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} dx dy < \infty.$$

The set $\{\psi_k\}$

$$\psi_k(z) = \pi^{-1/2}(k!)^{-1/2} z^k, \qquad k = 0, \pm 1, \pm 2, \ldots$$

is an orthonormal basis.

Home exercises.

1. Show that the function h defined in (3) satisfies the equation

$$h''(t) = h(t), \qquad h'(\pi) = h'(-\pi).$$

2. *(ex. 6.4.2 from AFr)* Let H_0 be a linear subspace of a Hilbert space H, spanned by a sequence $\{x_m\}$ of linearly independent elements. Show that there exist scalars λ_{mn} s.t. the sequence $\{y_m\}$ given by

$$y_m \sum_{n=1}^{m} \lambda_{mn} x_n$$

is orthonormal and it spans H_0. This process of passing from $\{x_m\}$ to $\{y_m\}$ is called the Gram-Schmidt process.

3. *(ex. 6.4.6 from AFr)* Show that an orthonormal sequence $\{\varphi_n\}$ is complete in $L^2(a, b)$ if

$$\sum_{n=1}^{\infty} \left(\int_a^x \varphi_n(t)\, dt \right)^2 = x - a$$

for all $x \in (a, b)$.

4. *(ex. 6.4.7 from AFr)* If $\{x_n\}$ is a complete orthonormal sequence in a Hilbert space H and if $\{y_n\}$ is an orthonormal sequence in H satisfying

$$\sum_{n=1}^{\infty} \|x_n - y_n\|^2 < 1,$$

then $\{y_n\}$ is also complete.