The norm of the Laplace transform

Problem.

Let L be the Laplace transform

$$g(s) = Lf(s) = \int_0^\infty f(t)e^{-st}\,dt$$

Then the operator $L : L^2(o,\infty) \to L^2(o,\infty)$ is bounded and

$$\|L\| \le \sqrt{\pi}.$$

Proof.

$$|g(s)|^2 = \Big(\int_0^\infty f(t)e^{-st}\,dt\Big)^2 = \Big(\int_0^\infty f(t)e^{-st/2}t^{1/4}e^{-st/2}t^{-1/4}\,dt\Big)^2$$

$$\le \int_0^\infty |f(t)|^2 e^{-st}t^{1/2}\,dt \int_0^\infty e^{-st}t^{1/2}\,dt$$

Now

$$\int_0^\infty e^{-st}t^{1/2}\,dt = s^{-1/2}\int_0^\infty e^{-u}u^{-1/2}\,du = \sqrt{\pi}s^{-1/2}.$$

Therefore

$$|g(s)|^2 \le \sqrt{\pi}s^{-1/2}\int_0^\infty |f(t)|^2 e^{-st}t^{1/2}\,dt.$$

Integrating this inequality we obtain

$$\|g\|_{L^2}^2 = \int_0^\infty |g(s)|^2\,ds \le \sqrt{\pi}\int_0^\infty\int_0^\infty |f(t)|^2 e^{-st}t^{1/2}s^{-1/2}dtds.$$

Since

$$\int_0^\infty e^{-st}t^{1/2}s^{-1/2}ds = \int_0^\infty e^{-u}u^{-1/2}\,du = \sqrt{\pi}$$

we finally have

$$\|g\|_{L^2}^2 \le \pi.$$