Homework 2 Due date December 5th

Due date December 5th, send it by e-mail at wojtek@math.kth.se or put it in my mailbox in the math department

Problem 1. (5 x 5 points) Describe the following as abelian groups:

$$\hom_{\mathbf{Z}}(\mathbf{Z},\mathbf{Z}/n) \quad \hom_{\mathbf{Z}}(\mathbf{Z}/m,\mathbf{Z}/n) \quad \hom_{\mathbf{Z}}(\mathbf{Z}/n,\mathbf{Z}) \quad \hom_{\mathbf{Z}}(\mathbf{Q},\mathbf{Z}) \quad \hom_{\mathbf{Z}}(\mathbf{Q},\mathbf{Q})$$

Problem 2. (10 points) Let $f : X \to Y$ be a morphism in a category C. Show that the following are equivalent:

- f is an isomorphism;
- for any object Z, the map of sets $f^* : \mathrm{mor}_C(Y,Z) \to \mathrm{mor}_C(X,Z)$ $(h \mapsto hf)$ is a bijection;

Problem 3.

(1) (10 points) Show that if M and N are simple left R modules, then any R-module homomorphism $f : M \to N$ is either zero or an isomorphism.

(2) (10 points) Show that if M is a simple left R-module, then the ring $\mathrm{End}_R(M)$ is a division ring, i.e. all non-zero elements have inverses.

Problem 4. Let $R = \mathbf{Z}[x]/(X^2+5)$.

(1) (5 points) Show that there is a unique ring homomorphism $\pi : R \to \mathbf{Z}/2$.

(2) (15 points) Let I be the kernel of the homomorphism $\pi : R \to \mathbf{Z}/2$. Recall that I^2 consists of finite sums of the form $\sum x_i y_i$, where x_i and y_i belong to I. Show that I^2 is the ideal of R generated by 2, I is projective R-module, and I is not free R-module.

Problem 5. (20 points) Let C be a category. Recall that a morphism $f : X \to Y$ is an epimorphism if for any object Z, the function $\mathrm{mor}_C(Y,Z) \ni h \mapsto hf \in \mathrm{mor}_C(X,Z)$ is 1 to 1. Prove that in the category of Groups a morphism $f : G \to H$ is an epimorphism if and only if it is onto.