Set 3

Problem 1. Let $f : M \to N$ and $g : N \to P$ be homomorphisms of left R-modules. Show that there is an exact sequence:

$$0 \to \ker(f) \to \ker(gf) \to \ker(g) \to \operatorname{coker}(f) \to \operatorname{coker}(gf) \to \operatorname{coker}(g) \to 0$$

Problem 2. Let R be a commutative ring and $r \in R$. Denote by S the multiplicative system $\{1, r, r^2, \ldots, r^n \ldots, \}$. Show that there is a ring isomorphisms between $R[S^{-1}]$ and $R[X]/(rX - 1)$.

Problem 3. Show that any finitely generated projective module is finitely presented.

Problem 4. Let $R = \mathbf{Z}[x]/(X^2 + 5)$.

(1) Show that there is a unique ring homomorphism $\pi : R \to \mathbf{Z}/2$.

(2) Let I be the kernel of the homomorphism $\pi : R \to \mathbf{Z}/2$. Recall that I^2 consists of finite sums of the form $\sum x_i y_i$, where x_i and y_i belong to I. Show that I^2 is the ideal of R generated by 2, I is projective R-module, and I is not free R-module.

Problem 5. A left R-module M is called noetherian if all its submodules are finitely generated.

(1) Let $0 \to M_0 \to M_1 \to M_2 \to 0$ be an exact sequence of left R-modules. Show that M_1 is noetherian if and only if, both M_0 and M_2 are noetherian.

(2) Assume that R is a noetherian left R-modules. Show that any finitely generated left R-module is also noetherian.

Problem 6. Let R be the set of 2×2 matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ such that a and b are real numbers and d is a rational number. Show that R with matrix addition and multiplication is a ring. Prove that R is a noetherian left R-module.

Problem 7.

(1) Show that if M and N are simple left R modules, then any R-module homomorphism $f : M \to N$ is either zero or an isomorphism.

(2) Show that if M is a simple left R-module, then the ring $\operatorname{End}_R(M)$ is a division ring, i.e. all non-zero elements have inverses.