Set 2

Problem 1. Let $0 \to M_0 \to M_1 \to M_2 \to 0$ and $0 \to N_0 \to N_1 \to N_2 \to 0$ be exact sequences of R-modules. Suppose that we have a commutative diagram of the form:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle f_0} & & \big\downarrow{\scriptstyle f_1} & & {\scriptstyle f_2}\big\downarrow & & \\ 0 & \longrightarrow & N_0 & \longrightarrow & N_1 & \longrightarrow & N_2 & \longrightarrow & 0 \end{array}$$

Show that if any two of f_0, f_1, f_2 are isomorphisms, then so is the third.

Problem 2. Consider the following commutative diagram of R-modules with exact rows:

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \big\downarrow{\scriptstyle f_1} & & \big\downarrow{\scriptstyle f_2} & & \big\downarrow{\scriptstyle f_3} & & \big\downarrow{\scriptstyle f_4} & & \big\downarrow{\scriptstyle f_5} \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

Show that if f_1, f_2, f_4, and f_5 are isomorphisms, then f_3 is also and isomorphism.

Problem 3. Give examples of exact sequences of abelian groups:

$$0 \to A_0 \to A_1 \to A_2 \to 0 \quad , \quad 0 \to B_0 \to B_1 \to B_2 \to 0$$

such that:

a. $A_0 \simeq B_0$, $A_1 \simeq B_1$, and $A_2 \not\simeq B_2$.
b. $A_0 \simeq B_0$, $A_1 \not\simeq B_1$, and $A_2 \simeq B_2$.
c. $A_0 \not\simeq B_0$, $A_1 \simeq B_1$, and $A_2 \simeq B_2$.

Problem 4. We say that a morphism $f : X \to Y$ is a retract of $g : A \to B$ if there is a commutative diagram of the form:

$$\begin{array}{ccccc} & & \overset{\mathrm{id}_X}{\frown} & & \\ X & \longrightarrow & A & \longrightarrow & X \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle f} \\ Y & \longrightarrow & B & \longrightarrow & Y \\ & & \underset{\mathrm{id}_Y}{\smile} & & \end{array}$$

Show that if g is an isomorphism, epimorphism, monomorphism of R-modules then so is any of its retract.

Problem 5. Describe the following as abelian groups:

$$\hom_{\mathbf{Z}}(\mathbf{Z}, \mathbf{Z}/n) \quad \hom_{\mathbf{Z}}(\mathbf{Z}/m, \mathbf{Z}/n) \quad \hom_{\mathbf{Z}}(\mathbf{Z}/n, \mathbf{Z}) \quad \hom_{\mathbf{Z}}(\mathbf{Q}, \mathbf{Z}) \quad \hom_{\mathbf{Z}}(\mathbf{Q}, \mathbf{Q})$$

Problem 6. Let $0 \to M_0 \xrightarrow{f} M_1 \xrightarrow{g} M_2 \to 0$ be an exact sequence of R-modules. Show that the following statements are equivalent:

a. There exists $s : M_2 \to M_1$ such that $gs = \mathrm{id}_{M_2}$.
b. There exists $t : M_2 \to M_1$ such that $tf = \mathrm{id}_{M_0}$.

c. The following is an exact sequence for any R-module N:

$$0 \to \hom_R(N, M_0) \xrightarrow{f_*} \hom_R(N, M_1) \xrightarrow{g_*} \hom_R(N, M_2) \to 0$$

d. The following is an exact sequence for any R-module N:

$$0 \to \hom_R(M_2, N) \xrightarrow{g^*} \hom_R(M_1, N) \xrightarrow{f_*} \hom_R(M_0, N) \to 0$$

e. There is an R-module homomorphism $t : M_2 \to M_1$ such that the homomorphism $f \oplus t : M_0 \oplus M_2 \to M_1$, $(a, b) \mapsto f(a) + t(b)$, is an isomorphism.