Set 1

Problem 1. Let C be a category. Recall that:

- a morphism $f : X \to Y$ is an isomorphism if there is $g : Y \to X$ such that $fg = \mathrm{id}_y$ and $gf = \mathrm{id}_X$.
- a morphism $f : X \to Y$ is a monomorphism if for any object Z, the function $\mathrm{mor}_C(Z, X) \ni h \mapsto fh \in \mathrm{mor}_C(Z, Y)$ is 1 to 1.
- a morphism $f : X \to Y$ is an epimorphism if for any object Z, the function $\mathrm{mor}_C(Y, Z) \ni h \mapsto hf \in \mathrm{mor}_C(X, Z)$ is 1 to 1.

(1) Show that in the categories $Sets$, $R - Mod$, and $Groups$ a morphism is an isomorphism if and only if it is a bijection.
(2) Show that in the categories $Sets$, $R - Mod$, and $Groups$ a morphism is a monomorphism if and only if it is 1 to 1.
(3) Show that in the categories $Sets$, $R - Mod$, and $Groups$ a morphism is an epimorphism if and only if it is onto.
(4) Show that in the category $Rings$, the morphism $\mathbf{Z} \to \mathbf{Q}$ is an epimorphism.
(5) Find an example of a category and a morphism which is both an epimorphism and a monomorphism but it is not an isomorphism.

Problem 2. Let $f : X \to Y$ be a morphism in C. Show that the following are equivalent:

(1) f is an isomorphism;
(2) for any object Z, the map of sets $f^* : \mathrm{mor}_C(Y, Z) \to \mathrm{mor}_C(X, Z)$ $(h \mapsto hf)$ is a bijection;
(3) for any object Z, the map of sets: $f_* : \mathrm{mor}_C(Z, X) \to \mathrm{mor}_C(Z, Y)$ $(h \mapsto fh)$ is a bijection.

Problem 3. Show how to represent a group as a category with a single object, morphisms identified with elements of the group and their composition with the group multiplication.

Problem 4. Let C be a category. Show how to define a new category C^{op} such that:

- objects of C^{op} are the same as objects of C;
- for any such objects X and Y, $\mathrm{mor}_{C^{op}}(X, Y) := \mathrm{mor}_C(Y, X)$.

You need to show how to compose morphisms in C^{op} so that the associativity relation $(fg)h = f(gh)$ holds, and find the identity morphisms.

Problem 5. Let us use the same symbol G to denote the category associated with a group G (see Problem 3). Show that the categories G and G^{op} are equivalent.

Problem 6. An object E in a category C is called initial if, for any Z, the set of morphisms $\mathrm{mor}_C(E, Z)$ consists of exactly one morphism. An object T in a category C is called terminal if, for any Z, the set of morphisms $\mathrm{mor}_C(Z, T)$ consists of exactly one morphism.

(1) Show that any two initial objects are isomorphic.
(2) Show that any two terminal objects are isomorphic.
(3) show that if E is initial (terminal) in C then E is terminal (initial) in C^{op}.
(4) Let $F : C \to D$ be an equivalence of categories. Show that if E is initial (terminal) in C, then so is $F(E)$ in D.

(5) Do the categories *Sets*, *Groups*, $R - Mod$, and *Rings* have initial and terminal objects? If so find them.

Problem 7. The symbol $a \leftarrow b \rightarrow c$ denotes the category with 3 objects $\{a, b, c\}$ and two non-identity morphisms $b \rightarrow a$ and $b \rightarrow c$. The symbol $a \rightarrow b \leftarrow c$ denotes the category with 3 objects $\{a, b, c\}$ and two non-identity morphisms $a \rightarrow b$ and $c \rightarrow b$.

(1) Show that the categories $a \leftarrow b \rightarrow c$ and $a \rightarrow b \leftarrow c$ are not isomorphic.
(2) Show that $(a \leftarrow b \rightarrow c)^{op}$ and $a \rightarrow b \leftarrow c$ are isomorphic.
(3) give an example of a category C for which C and C^{op} are not equivalent.

Problem 8. An object U of a category C is called a generator if it has the following property: for any different morphisms $f : X \rightarrow Y$ and $g : X \rightarrow Y$, there is a morphism $h : U \rightarrow X$ such that $fh \neq gh$. Prove that the category *Groups* has a generator.

Problem 9. Let G be a group. Recall that the center $Z(G)$ of G consists of these elements $a \in G$ such that for any $g \in G$, $ag = ga$. The center $Z(G)$ is a subgroup of G. Show that there is no functor F : *Groups* $\rightarrow$ *Groups* such that for any G, $F(G)$ is isomorphic to $Z(G)$.

Tip: Observe that $Z(\Sigma_3) = 0$. Construct group homomorphisms: $\mathbf{Z}/2 \xrightarrow{f} \Sigma_3 \xrightarrow{g} \mathbf{Z}/2$ such that $gf = \text{id}$. Apply the functor F to these morphisms and find a contradiction with the facts that $F(\text{id}) = \text{id}$ and $F(\mathbf{Z}/2) = \mathbf{Z}/2 \neq 0$.