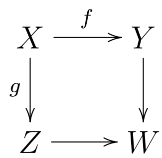
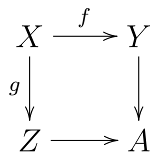
Lecture on February 28

1. Let $Y \xleftarrow{f} X \xrightarrow{g} Z$ be continuous maps. Consider the following relation on $Y \coprod Z$: $f(x) \sim g(x)$ for $x \in X$. This may not be an equivalence relation, but it generates one, denoted by the same symbol $\sim$. Define $W := (Y \coprod Z)/\sim$. Let $\pi : Y \times Z \to W$ be the map that assigns to an element its equivalence class. Consider the smallest topology on W for which π is continuous. That means that U in W is open if and only if $\pi^{-1}(U)$ is open in $Y \coprod Z$.

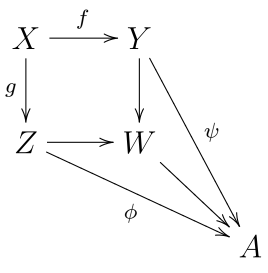
Not that te compositions $Y \subset Y \coprod Z \xrightarrow{\pi} W$ and $Z \subset Y \coprod Z \xrightarrow{\pi} W$ are continuous. Moreover these maps fit into the following commutative square:



For any map $W \to A$, the compositions $Y \to W \to A$ and $Z \to W \to A$ are continuos. Moreover they fit into the following commutative diagram:



On the other hand, for any maps $\psi : Y \to A$ and $\phi : Z \to A$ such that $\psi f = \phi g$ there is a unique map $W \to A$ for which the following diagram commutes:



This means that the set of continuos maps $W \to A$ is in bijection with the set of pairs of continuous maps $\psi : Y \to A$ and $\phi : A \to A$ such that $\psi f = \phi g$.

A commutative square:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow \\ Z & \longrightarrow & A \end{array}$$

is called a push-out if the induced map $W \to A$ is a homeomorphism.

2. Assume that the following is a push-out square:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow \\ Z & \longrightarrow & A \end{array}$$

Then for any W, the following is also a push-out square:

$$\begin{array}{ccc} X \times W & \xrightarrow{f \times \mathrm{id}_W} & Y \times W \\ {\scriptstyle g \times \mathrm{id}_W}\downarrow & & \downarrow \\ Z \times W & \longrightarrow & A \times W \end{array}$$

3. Let $0 < a < 1$ and X be a space. The following are push-out squares:

$$\begin{array}{ccc} \{a\} & \hookrightarrow & [0,a] \\ \downarrow & & \downarrow \\ [a,1] & \hookrightarrow & [0,1] \end{array} \qquad \begin{array}{ccc} X \times \{a\} & \hookrightarrow & X \times [0,a] \\ \downarrow & & \downarrow \\ X \times [a,1] & \hookrightarrow & X \times [0,1] \end{array}$$

Thus to define a continuous map $X \times [0,1] \to Y$ it is enough to define a pair of continuous maps $X \times [a,1] \to Y$ and $X \times [0,a] \to Y$ which agree on $X \times \{a\}$.