Lecture on February 21

1. A finite product of compact spaces is a compact space.

2. Recall that $S^0 = \{-1, 1\} \subset \mathbf{R}$ is a discrete space. Let $f : X \to S^0$ be a continuous maps. There are three possibilities: a) the image of f is -1, b) the image of f is 1, c) f is an onto map. A space X is called connected if there is no onto map $X \to S^0$. Thus X is connected if the set of continuos maps $\mathrm{map}(X, S^0)$ consists of two elements.

3. Subspace $A \subset X$ and $B \subset X$ are called separated if $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$. The following are equivalent:

(1) X is connected.
(2) The only open and closed subsets of X are X and $\emptyset$.
(3) If $X = A \cup B$ and $A \neq \emptyset \neq B$, then A and B are not separated (either $\overline{A} \cap B \neq \emptyset$ or $A \cap \overline{B} \neq \emptyset$).
(4) If $X = A \cup B$ and $A \neq \emptyset \neq B$, then $\overline{A} \cap \overline{B} \neq \emptyset$.

4. Any interval (open, closed, one side open) are connected spaces. Any product of connected spaces is a connected space.

5. X is called path-connected if any map $S^0 \to X$ can be expressed as a composition: $S^0 \hookrightarrow D^1 \to X$. Equivalently for any two points $x_0 \in X$ and $x_1 \in X$, there is a map $\omega : [0, 1] \to X$ (a path) such that $\omega(0) = x_0$ and $\omega(1) = x_1$.

6. Any path connected space is a connected space.