Part of the lecture on February 14

1. A space is *compact* if for any family of open subspaces $\{U_i\}_{i\in I}$ such that $X = \cup_{i\in I} U_i$ there are finitely many $U_{i_1}, \ldots, U_{i_k}$ for which $X = U_{i_1} \cup \cdots \cup U_{i_k}$.

Let $A \subset X$ be a subspace. A is a compact space if and only if for any family $\{U_i\}_{i\in I}$ of open subsets in X such that $A \subset \cup_{i\in I} U_i$, there are are finitely many $U_{i_1}, \ldots, U_{i_k}$ for which $A \subset U_{i_1} \cup \cdots \cup U_{i_k}$.

Let $f : X \to Y$ be a continuous map. If $A \subset X$ is compact, then so is $f(A)$.

If X is compact, then so is any closed subset of X.

2. A space is Hausdorff if for any two points $x, y \in X$, there are open sets U and V such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

Any metric space is Hausdorff.

If X is Hausdorff, then any compact subspace of X is closed.

If X is metric and compact space, then any sequence $\{x_i\}_{i>0}$ of elements in X has a convergent subsequence.

Let X be metric and compact. Then, for any family of open subspaces $\{U_i\}_{i\in I}$ for which $X = \cup_{i\in I} U_i$, there is a number $r > 0$ with the following property: for any $x \in X$, there is $i \in I$ such that $B(x, r) \subset U_i$. Here $B(x, r) = \{y \in X \mid d(x, y) < r\}$.

3. Let $f : X \to Y$ be a continuous map. Assume that X is compact, Y is Hausdorff, f is 1-1 and f is onto. Then f is a homeomorphism. To see this we need to show that f takes closed sets to closed sets (in this way f^{-1} would be continuous). Since X is compact, then so is any of its closed subsets. Their images must therefore be also compact in Y. As Y is Hausdorff, these compact sets are also closed.

4. The interval $[0, 1] \subset \mathbf{R}$ is a compact space.

5. Let X and Y be topological spaces. Consider the product $X \times Y$ with the topology induced by the base:

$$\{U \times V \mid U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$$

A subset $S \subset X \times Y$ is open if $S = \cup(U_i \times V_j)$ for some open subsets $U_i \subset X$ and $V_j \subset Y$. We call this topological space the product of X and Y. This space has the following properties:

(1) The projections $\mathrm{pr}_1 : X \times Y \to X$, $(x, y) \mapsto x$, and $\mathrm{pr}_2 : X \times Y \to Y$, $(x, y) \mapsto y$, are continuous maps.
(2) For any two continuous maps $f : Z \to X$ and $g : Z \to Y$ there is a unique continuous map $h : Z \to X \times Y$ for which the

following diagram commutes: