Homework 1' Due date March 6th

Due date March 6th, send it by e-mail at wojtek@math.kth.se or put it in my mailbox in the math department

Problem 1. Show that if $n > 0$, then $S^n = \{x \in \mathbf{R}^{n+1} \mid |x| = 1\}$ is a connected space.

Let $\{X_i\}_{i>0}$ be a family of topological spaces. Define a topology on $\prod_{i>0} X_i$ by a base consisting of subsets of the form: $\{(x_i) \in \prod_{i>0} X_i \mid x_{i_l} \in U_l\}$ for some finite number of open subsets $U_l \subset X_{i_l}, l = 1, \dots k$.

Problem 2.

a. Prove that the functions $\mathrm{pr}_k : \prod_{i>0} X_i \to X_k$, $(x_i) \mapsto x_k$ are continuous.
b. Let $\{f_k : Z \to X_k\}_{k>0}$, be a set of maps with the following property: for any set of continuous maps $\{g_k : Y \to X_k\}_{k>0}$, there is a unique continuous map $h : Y \to Z$ such that $f_k h = g_k$, for any k. Prove that Z is homeomorphic to $\prod_{i>0} X_i$.

Solve either problem 3 or problem 4.

Problem 3. Show that if, for $i > 0$, X_i are compact, then $\prod_{i>0} X_i$ is compact.

Problem 4.

(a) We say that a family of closed subsets $\{A_l\}_{l\in I}$ in X has a finite intersection property if for any **finite** subset $J \subset I$, the intersection $\bigcap_{j\in J} A_j$ is not empty.

Prove that X is compact if and only if for any family of closed subsets $\{A_l\}_{l\in I}$ in X with a finite intersection property, the intersection $\bigcap_{l\in I} A_l$ is not empty.

(b) Show, using Kuratowski-Zorn's lemma, that there exists a family $\overline{F} = \{B_s\}_{s\in S}$ of closed subset of X such that
- $F \subset \overline{F}$,
- $\overline{F}$ has a finite intersection property,
- if B is a closed subset of X such that for any finite $J \subset S$, $B \cap \bigcap_{j\in J} B_j \neq \emptyset$, then $B \in \overline{F}$.

(c) Use parts (a) and (b) to show that if, for $i > 0$, X_i are compact, then $\prod_{i>0} X_i$ is compact. Take a family of closed subsets $\mathcal{F} = \{A_l\}_{l\in I}$ in $\prod_{i>0} X_i$ that has a finite intersection property. Consider $\overline{\mathcal{F}}$. Show that $\{\overline{\mathrm{pr}_k(A)} \mid A \in \overline{\mathcal{F}}\}$ is a family of closed subset in X_k that has a finite intersection property. Take $x_k \in \bigcap_{A\in\overline{\mathcal{F}}} \overline{\mathrm{pr}_k(A)}$ (use compactness of X_k). Show that $(x_k) \in \bigcap_{i\in I} A_i$.