Matrix Groups - Homework set 9

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Assignment 1. *Let $\phi \in \mathrm{GL}_{n+1}(\mathbb{C})$, and let $L \subseteq \mathbb{C}^{n+1}$ be a vector subspace of dimension 1. Show that $\phi(L)$ is also a vector subspace of dimension 1.*

Thus ϕ induces a map of sets $F_\phi : \mathbf{P}^n \longrightarrow \mathbf{P}^n$ in the following way: Let $p \in \mathbf{P}^n$ be a point, and let $L_p = C(p) \subseteq \mathbb{C}^{n+1}$ be the associated line through the origin. Then we let $F_\phi(p) \in \mathbf{P}^n$ be the point corresponding to the line $\phi(L_p) \subseteq \mathbb{C}^{n+1}$.

Show that $F_\phi : \mathbf{P}^n \longrightarrow \mathbf{P}^n$ is an isomorphism in the category of varieties.

Proof. If $L \subseteq \mathbb{C}^{n+1}$ is a linear subspace of dimension one, then L is generated by a non-zero vector $v \in \mathbb{C}^{n+1}$. The space $\phi(L)$ will then be generated by the vector $\phi(v)$ which is non-zero since ϕ is invertible.

We will now show that F_ϕ is a homeomorphism. First of all, we note that $F_\phi^{-1} = F_{\phi^{-1}}$ as maps of sets. Thus if we can show that F_ϕ takes closed sets to closed sets, we are done. Let $V(\mathfrak{a}) \subseteq \mathbf{P}^n$ be a closed set, where $\mathfrak{a} \subseteq \mathbb{C}[x_0, \ldots, x_n]$ is a homogeneous radical ideal. Then

$$F_\phi(V(\mathfrak{a})) = \{F_\phi(p) \in \mathbf{P}^n : \ f(p) = 0 \ \forall f \in \mathfrak{a}\} =$$

$$= \left\{ q \in \mathbf{P}^n : \ f(F_\phi^{-1}(q)) = 0 \ \forall f \in \mathfrak{a} \right\}.$$

For every homogeneous polynomial $f \in \mathfrak{a}$ we define a homogeneous polynomial of degree $\deg f$ as

$$\widetilde{f}(x_0, \ldots, x_n) = f(\phi^{-1}(x_0, \ldots, x_n)).$$

Then

$$F_\phi(V(\mathfrak{a})) = \left\{ q \in \mathbf{P}^n : \ \widetilde{f}(q) = 0 \ \forall f \in \mathfrak{a} \right\} = V(\mathfrak{b}),$$

where $\mathfrak{b} \subseteq \mathbb{C}[x_0, \ldots, x_n]$ is the ideal generated by the polynomials $\left\{ \widetilde{f} : \ f \in \mathfrak{a} \right\}$. This shows that F_ϕ maps closed sets to closed sets, and thus F_ϕ is a homeomorphism.

To show that F_ϕ is an isomorphism, let $U \subseteq \mathbf{P}^n$ be an open set, and let $h = f/g \in \mathcal{O}(U)$ be a regular function, where $f, g \in \mathbb{C}[x_0, \ldots, x_n]$ are homogeneous polynomials of the same degree. Then we must check that $h \circ F_\phi$ is a regular function on $F_\phi^{-1}(U)$. But $h \circ F_\phi$ is represented by the rational function

$$\frac{f(\phi(x_0, \ldots, x_n))}{g(\phi(x_0, \ldots, x_n))}$$

which is a quotient of polynomials of the same degree, where the denominator never vanishes on $F_\phi^{-1}(U) = F_{\phi^{-1}}(U)$. This shows that $h \circ F_\phi$ is a regular function, so F_ϕ is a morphism and thus an isomorphism. $\square$

Definition 1. Two projective varieties $X, Y \subseteq \mathbf{P}^n$ are said to be **linearly isomorphic** if there is some $\phi \in \mathrm{GL}_{n+1}(\mathbb{C})$ such that $Y = F_\phi(X)$.

Assignment 2. *Let $f \in \mathbb{C}[x_0, \dots, x_n]$ be a homogeneous polynomial of degree 2. The corresponding variety $V(f) \subseteq \mathbf{P}^n$ is called a **quadric**. Define a function $Q_f : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ by*

$$Q_f(u, v) = \frac{f(u+v) - f(u) - f(v)}{2}.$$

(a) *Show that Q_f is a bilinear symmetric form.*

(b) *Let Q_f^* denote the linear map corresponding to Q_f. Define the **rank** of the polynomial f as $\mathrm{rk}(f) = \mathrm{rank}(Q_f^*)$. Show that two quadrics $V(f_1), V(f_2) \subseteq \mathbf{P}^n$ are linearly isomorphic if and only if $\mathrm{rk}(f_1) = \mathrm{rk}(f_2)$.*

(c) *By the above we have that a quadric $V(f) \subseteq \mathbf{P}^n$ is reducible (irreducible) iff $\mathrm{rk}(f) = \mathrm{rk}(g)$ where $V(g) \subseteq \mathbf{P}^n$ is reducible (irreducible). Using this, determine for a given integer $i \geq 1$ whether the quadrics of rank i in $\mathbf{P}^n$ are reducible or irreducible.*

Proof of (a). Let $x = (x_0, \dots, x_n)$ and $y = (y_0, \dots, y_n)$ be two sets of variables. The polynomial f is of the form

$$f(x) = \sum_{i \leq j} \alpha_{ij} x_i x_j = x^T A x$$

where A is the matrix defined by

$$(A)_{ij} = \begin{cases} \alpha_{ii} \text{ if } i = j, \\ \frac{\alpha_{ij}}{2} \text{ if } i < j, \\ \frac{\alpha_{ji}}{2} \text{ if } j < i. \end{cases}$$

We note that A is symmetric. We have

$$Q_f(x, y) = \frac{1}{2}((x+y)^T A(x+y) - x^T A x - y^T A y) = \frac{1}{2}(x^T A y + y^T A x) =$$

$$= \frac{1}{2} y^T (A + A^T) x = y^T A x.$$

This shows that Q_f is a bilinear symmetric form. $\square$

Proof of (b). Let $f_1, f_2 \in \mathbb{C}[x_0, \dots, x_n]$ be two homogeneous quadratic polynomials, and let $A_1 = Q_{f_1}^*$ and $A_2 = Q_{f_2}^*$. Assume first that $V(f_1)$ and $V(f_2)$ are linearly isomorphic. Thus there is some $\phi \in \mathrm{GL}_{n+1}(\mathbb{C})$ such that $V(f_2) = F_\phi(V(f_1))$. From Assignment 1 we have that $V(f_2) = V(\widetilde{f_1})$, where

$$\widetilde{f_1}(x_0, \dots, x_n) = f_1(\phi^{-1}(x_0, \dots, x_n)).$$

Thus $\sqrt{(f_2)} = \sqrt{(\widetilde{f_1})}$, and since f_2 and $\widetilde{f_1}$ are both of degree two, this means that $f_2 = \alpha\widetilde{f_1}$ for some non-zero $\alpha \in \mathbb{C}$.

Let B be the matrix corresponding to the linear map ϕ^{-1}. Then

$$f_2(x) = x^T A_2 x = \alpha(Bx)^T A_1(Bx) = x^T(\alpha B^T A_1 B)x.$$

Therefore $A_2 = \alpha B^T A_1 B$ where B is invertible. This shows that $\operatorname{rank}(A_1) = \operatorname{rank}(A_2)$, and so $\operatorname{rk}(f_1) = \operatorname{rk}(f_2)$.

Next assume that $\operatorname{rk}(f_1) = \operatorname{rk}(f_2)$. Then, since A_1 and A_2 are symmetric of the same rank, we can by Paragraph 1.7.8 in Boij/Laksov find an invertible matrix S such that $A_2 = S^T A_1 S$. Thus

$$f_2(x) = x^T A_2 x = x^T S^T A_1 S x = (Sx)A_1(Sx).$$

If we let ϕ^{-1} be the linear map corresponding to the matrix S, we have that $f_2 = \widetilde{f_1}$, and so $V(f_2) = F_\phi(V(f_1))$. This shows that $V(f_1)$ and $V(f_2)$ are linearly isomorphic. □

Proof of (c). For $i = 1$ one has that all quadratic polynomials whose associated matrix has rank 1 must be a square of a linear polynomial. Thus the corresponding variety is a linear variety, and thus irreducible.

For $i = 2$ we consider the polynomial $f_2 = x_0 x_1$. This corresponds to a reducible variety $V(f_2) = V(x_0) \cup V(x_1)$, and the corresponding $(n+1)\times(n+1)$-matrix is

$$Q^*_{f_2} = \begin{pmatrix} 0 & 1/2 & 0 & \cdots & 0 \\ 1/2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which has rank 2. Thus all quadrics of rank 2 in $\mathbf{P}^n$ are reducible.

For $i = 3$, consider the polynomial $f_3 = x_0^2 + x_1^2 + x_2^2$. By viewing f_3 as a polynomial in $(\mathbb{C}[x_0, x_1])[x_2]$ and choosing $p = x_0 + ix_1 \in \mathbb{C}[x_0, x_1]$ we have by the Eisenstein criterion that f_3 is irreducible. Its corresponding matrix is a diagonal matrix with 1 as first three elements and 0 as the rest. Thus the matrix has rank 3, so all polynomials of rank 3 are irreducible.

For $i > 3$ we choose the polynomial $f_i = x_0^2 + \ldots + x_i^2$. By induction the polynomial $x_0^2 + \ldots + x_{i-1}^2$ is irreducible, and so we can use Eisenstein criterion again to conclude that f_i is irreducible. By the same argument as the case $i = 3$, its corresponding matrix will have rank i.

The conclusion is that a quadric $V(f) \subseteq \mathbf{P}^n$ is reducible iff $\operatorname{rk}(f) = 2$. □

Assignment 3. *Consider the map* $d : \mathrm{sl}_2(\mathbb{C}) \longrightarrow \mathbb{C}$ *defined by*

$$\mathrm{sl}_2(\mathbb{C}) \ni A \longmapsto d(A) = \det A \in \mathbb{C}.$$

We identify $\mathrm{sl}_2(\mathbb{C})$ *with* $\mathbb{C}^3$ *by the isomorphism*

$$\mathrm{sl}_2(\mathbb{C}) \ni \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \longleftrightarrow (x, y, z) \in \mathbb{C}^3.$$

For each $t \in \mathbb{C}$ *define* $X_t = d^{-1}(t) \subseteq \mathbb{C}^3$.

(a) *Show that* $X_t \subseteq \mathbb{C}^3$ *is an irreducible affine variety for each* $t \in \mathbb{C}$.

(b) *We let* $\mathbb{C}[x, y, z, w]$ *be the homogeneous coordinate ring of* $\mathbf{P}^3$. *Then we identify* $\mathbb{C}^3$ *with the open subset* $D(w) = \mathbf{P}^3 - V(w) \subseteq \mathbf{P}^3$.

Let $t \in \mathbb{C}$. *Show that there is a unique quadratic homogeneous polynomial* $f_t \in \mathbb{C}[x, y, z, w]$ *such that* $V(f_t) \cap D(w) \cong X_t \subseteq \mathbb{C}^3$.

(c) *Show that* $\operatorname{rk}(f_t) = 4$ *if* $t \neq 0$ *and that* $\operatorname{rk}(f_0) = 3$.

Proof of (a). We have for each $t \in \mathbb{C}$, that $X_t = V(x^2 + yz + t)$. But we can use for instance the Eisenstein criterion to deduce that the polynomial $x^2 + yz + t$ is irreducible for each $t \in \mathbb{C}$. Thus the corresponding variety must also be irreducible. □

Proof of (b). We let $f_t \in \mathbb{C}[x, y, z, w]$ be the homogeneous polynomial

$$f_t = x^2 + yz + tw^2.$$

We now have an isomorphism between $V(f_t) \cap D(w)$ and X_t defined by

$$\begin{array}{ccc} V(f_t) \cap D(w) & \longrightarrow & X_t \\ (a_0 : a_1 : a_2 : a_3) & \longrightarrow & (a_0/a_3, a_1/a_3, a_2/a_3) \\ (b_0 : b_1 : b_2 : 1) & \longleftarrow & (b_0, b_1, b_2) \end{array}$$

□

Proof of (c). Since we have

$$f_t(x, y, z, w) = x^2 + yz + tw^2 = \begin{pmatrix} x & y & z & w \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & t \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

we have by Assignment 2(a) that

$$Q_{f_t}^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & t \end{pmatrix}.$$

Thus it is clear by inspection that the rank of the matrix $Q_{f_t}^*$ is 4 if $t \neq 0$, and that the rank is 3 if $t = 0$. □