KTH Matematik

5B1456, Matrix Algebra
Final assignment
due Dec. 6, 2004

(1) Let $\phi \in GL_{n+1}(\mathbb{C})$. Show that if $L \subset \mathbb{C}^{n+1}$ is a vector subspace of dimension 1, then $\phi(L)$ is a vector subspace of the same dimension. Prove that the assignment:

$$\mathbb{P}^n_{\mathbb{C}} \ni L \mapsto \overline{\phi}(L) := \phi(L) \in \mathbb{P}^n_{\mathbb{C}}$$

is a well defined continuous map, which is an ismorphism of topological spaces.

We say that two projective varieties $V(I), V(J)$ are *linearly isomorphic* if there is $\phi \in GL_{n+1}(\mathbb{C})$ such that $\overline{\phi}(V(I)) = V(J)$.

Let $Q(x_0, ..., x_n) \in \mathbb{C}[x_0, ..., x_n]$ be a homogeneous polynomial of degree 2. The projective variety $V(Q)$ is called a quadric in $\mathbb{P}^n$. Define a function $\mathcal{Q} : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \to \mathbb{C}$ by the formula:

$$\mathcal{Q}(v, w) = \frac{Q(v + w) - Q(v) - Q(w)}{2}$$

(2) Show that $\mathcal{Q}$ is a biliner symmetric form.
(3) Let $\mathcal{Q}^*$ be the associated linear map. Define the rank of Q as $\text{rk}(Q) = \text{ rank } (\mathcal{Q}^*)$. Show that two quadrics $V(Q_1)$ and $V(Q_2)$ are linearly isomorphic if and only if $rk(Q_1) = rk(Q_2)$.
(4) For each $n = 1, 2, 3$ give an example of rank for which $V(Q)$ is REDUCIBLE.

Consider the determinant map $\det : sl_2(\mathbb{C}) \to \mathbb{C}$. Let $t \in \mathbb{C}$. Define $V_t := \det^{-1}(t)$ (the preimage of t).

(5) Identify $sl_2(\mathbb{C})$ with $\mathbb{C}^3$ via the isomorhism: $\begin{pmatrix} x & y \\ z & -x \end{pmatrix} \mapsto (x, y, z)$. Show that, for every $t \in \mathbb{C}$, V_t is an irreducible affine variety in $\mathbb{C}^3$.

Consider the inclusion $\mathbb{C}^3 \subset \mathbb{P}^3$ given by $(z_1, z_2, z_3) \mapsto$ the line in $\mathbb{C}^4$ through $(z_1, z_2, z_3, 1)$.

(6) Show that there is a unique quadric $V(Q_t) \subset \mathbb{P}^3$ such that $V_t = V(Q_t) \cap \mathbb{C}^3$.
(7) Show that $rk(Q_t) = 4$ if $t \neq 0$ and that $rk(Q_0) = 3$.

Complemementary problem.

Two affine varieties $V(I)$ and $V(J)$ in $\mathbb{C}^n$ are said to be linearly isomorphic if there is a linear map $\alpha \in GL_n(\mathbb{C})$ and a translation α_P of $\mathbb{C}^n$ such that $(\alpha \circ \alpha_P)(V(I)) = V(J)$.

(A) Show that for every $t \neq 0$ V_t is linearly isomorphic to V_1 but it is not linearly isomorphic to V_0.