KTH Matematik

5B1456, Matrix Algebra
Assignment 8
due Nov. 22, 2004

Let K be an algebraically closed field.

(1) Show that:
  (a) For any two subsets of A_K^n, X and Y, if $X \subset Y$ then $I(Y) \subset I(X)$.
  (b) $I((a_1, ..., a_n)) = (x_1 - a_1, ..., x_n - a_m)$.
  (c) $J \subseteq I(V(J))$, for any ideal $J \subset K[x_1, ..., x_n]$.
  (d) $V(I(V(J)) = V(J)$ for any ideal $J \subset K[x_1, ..., x_n]$.

(2) Recall that:
- a domain R is a *unique factorization domain* (UFD) if every non-zero element can be factored uniquely, up to units and the order of the elements, into irreducible elements.
- If R is a UFD then $R[x_1, ..., x_n]$ is a UFD. The quotient field is usually denoted by $R(x_1, ..., x_n)$.
- (Gauss Lemma) If R is a UFD with quotient field Q then every irreducible element in $R[x]$ is still irreducible when considered as element in $Q[x]$.
- A *principal ideal domain* (PID) is an integral domain such that every ideal is principal (generated by one element). The main example of a PID is $K[x]$, where K is a field.

A good reference for the statements above is:
A first course in abstract algebra, by J. Fraleich, Ch. 45.

  (a) Let f, g be polynomials in $K[x, y]$ with no common factors. Prove that
$$(f, g) = (1) \text{ in } K(x)[y].$$
  (b) Deduce from (a) that $V(f, g) = V(f) \cap V(g)$ is a finite set of points.
  (c) Prove that the irreducible Zariski-closed subsets of A_K^2 are:
  $A_K^2, \emptyset,$ finite sets of points or $V(f)$ where f is an irreducible polynomial.
  (d) Deduce that $\dim(A_K^2) = 2$.