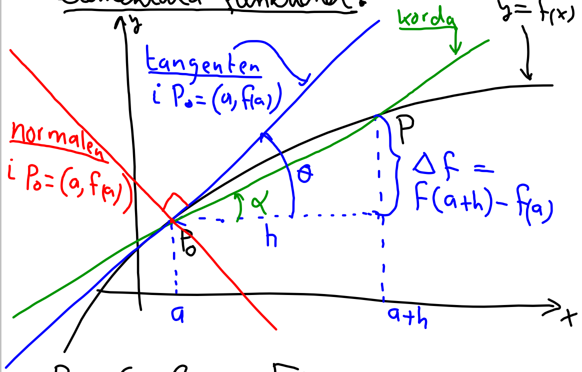
Förel nr 6: Om derivatan – Derivatan av elementära funktioner.



$P_0 = (a, f(a))$ Fixeras

Kordan $\overline{P_0P}$ har Lutningen

$$\tan\alpha = \frac{f(a+h)-f(a)}{h} = \frac{\Delta f}{h}$$

$h \to 0 \Longrightarrow$ (1) $\overline{P_0P}$ = kordan $\longrightarrow$ tangenten

(2) $\tan\alpha \longrightarrow \tan(\theta)$

$$\therefore \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \text{tal} = \tan\theta = f'(a)$$

= Lutningen till den linje som tangerar kurvan i $P_0 = (a, f(a))$

Def 1 f säges vara deriverbar i $x=a$ om $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = A =$ ETT TAL $= f'(a)$

Ekvationen till tangenten i $P_0 = (a, f(a))$ ges

$$y - f(a) = f'(a)(x-a)$$

normalen i $P_0 = (a, f(a))$ ges

$$y - f(a) = -\frac{1}{f'(a)}(x-a)$$

olika sätt att skriva derivatan

$$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} = \left[\begin{array}{l} x = a+h \\ h = x-a \end{array} \right] =$$

$$= \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

Även sätt $a = x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

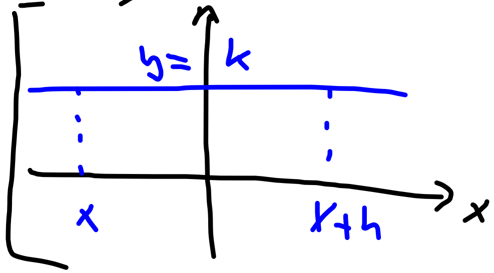
$$f'(x) = \frac{df}{dx} = Df$$

t. ex andra derivatan

$$f''(x) = \frac{d^2 f}{dx^2} = D^2 f$$

EX 1 $\quad f(x) = k = \text{konstant} \Longrightarrow f'(x) = 0$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} =$$



för $h \neq 0$

$$\frac{f(x+h) - f(x)}{h} = \frac{k-k}{h} = \frac{0}{h} = 0$$

$$\therefore \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 0$$

EX 2 $\quad f(x) = x \Longrightarrow f'(x) = 1$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$h \neq 0$, $\quad \dfrac{f(x+h) - f(x)}{h} = \dfrac{x+h-x}{h} = \dfrac{h}{h} = 1$

$$\therefore \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 1$$

Ex3 $f(x)=\sqrt{x} \implies f'(x)=\frac{1}{2\sqrt{x}} \quad x>0$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$h\neq 0$, $\frac{\sqrt{x+h}-\sqrt{x}}{h}=\left[\begin{array}{l}(a-b)(a+b)\\ = a^2-b^2\end{array}\right]=$

$$\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)=\frac{(x+h)-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}=$$

$$=\frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$\lim_{h\to\infty}\frac{\sqrt{x+h}-\sqrt{x}}{h}=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}=\frac{1}{\sqrt{x+0}+\sqrt{x}}$$

$$=\frac{1}{2\sqrt{x}}.$$

Sats

$$\boxed{\frac{d}{dx}\left[x^{\alpha}\right]=\alpha x^{\alpha-1} \text{ där } x \text{ är definierad}}$$

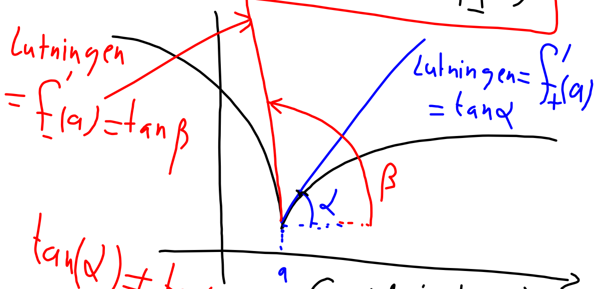
<u>Def 2</u> 1. f säges vara deriverbar från höger till $x=a$ om

$$\lim_{x\to a+} \frac{f(x)-f(a)}{x-a} = f_+'(a)$$

2. f säges vara deriverbar från vänster till $x=a$ om

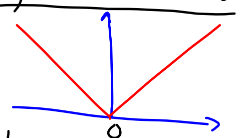
$$\lim_{x\to a-} \frac{f(x)-f(a)}{x-a} = f_-'(a)$$

$$\boxed{f'(a) \text{ finns} \Leftrightarrow f_+'(a) = f_-'(a)}$$



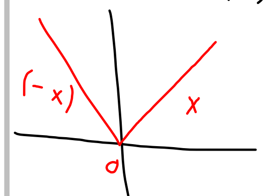
$\tan(\alpha) \neq \tan(\beta)$, f Ej deriverbar i $x=a$

<u>EX 1</u> $f(x) = |x|$



$$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{|x|-0}{x-0} = \lim_{x\to 0} \frac{|x|}{x}$$

$x \neq 0$ $\quad \frac{|x|}{x} = \begin{cases} \frac{x}{x}, & x>0 \\ -\frac{x}{x}, & x<0 \end{cases} = \begin{cases} 1, & x>0 \\ -1, & x<0 \end{cases}$



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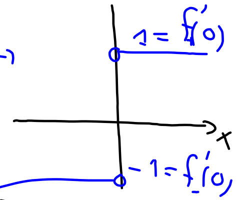


f Ej deriverbar i $x=0$

Sambandet mellan kontinuitet – deriverbarhet

Ur Ex 1 avs $f(x)=|x| \Rightarrow f'(0)$ finns inte

$f(x)=|x|$ kontin. $\not\Longrightarrow$ f deriverbar

$$\boxed{\text{Men Om } f \text{ deriverbar i } x=a \Longrightarrow f \text{ kontin. i } x=a}$$

Motivering:

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a) \text{ finns} \Rightarrow \lim_{x\to a} f(x) = f(a)$$

$x \neq a$

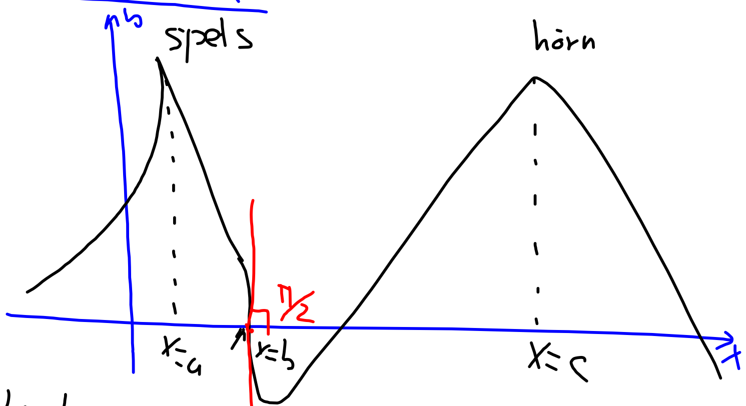
$$f(x)-f(a) = \frac{f(x)-f(a)}{(x-a)}(x-a)$$

$$\lim_{x\to a}\left[f(x)-f(a)\right] = \underbrace{\left(\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right)}_{f'(a)}\underbrace{\left(\lim_{x\to a}(x-a)\right)}_{=0}$$

$$\lim_{x\to a}\left[f(x)-f(a)\right] = 0 \Rightarrow \lim_{x\to a} f(x) = \lim_{x\to a} f(a) = f(a)$$

V.S.V

Vilka funktioner som inte är deriverbara



Kontinuerliga fkner = sammanhängande graf

Deriverbara fkner = —— // —— och slät graf (kan åka skridsko)

EX visa att f

EXA om $f(x) = \sin x \Longrightarrow f'(x) = \cos x$

[tillämpning av $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$]

$$\frac{d}{dx}(\sin x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} = \left[\begin{array}{l}\sin(x+h) = \\ \sin x \cosh + \\ \cos x \cdot \sinh\end{array}\right] =$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} =$$

$$= \lim_{h \to 0} \left(\sin x \frac{(\cos h - 1)}{h} + \cos x \frac{\sin h}{h} \right)$$

$$= \lim_{h \to 0} \underbrace{\sin x \left(\frac{\cos h - 1}{h} \right)}_{\cos h = 1 - 2\sin^2\left(\frac{h}{2}\right)} + \cos x \underbrace{\lim_{h \to 0} \frac{\sin h}{h}}_{= 1}$$

$$= \sin x \left(\lim_{h \to 0} \frac{\cancel{1} - 2\sin^2\left(\frac{h}{2}\right) \cancel{- 1}}{h} \right) + \cos x =$$

$$= \sin x \left(\lim_{h \to 0} - \sin\left(\frac{h}{2}\right) \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \right) + \cos x =$$

$$= -\sin x \underbrace{\lim_{h \to 0} \sin\left(\frac{h}{2}\right)}_{= 0} \underbrace{\lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right)}{h/2}}_{= 1} + \cos x$$

$$= \cos x$$

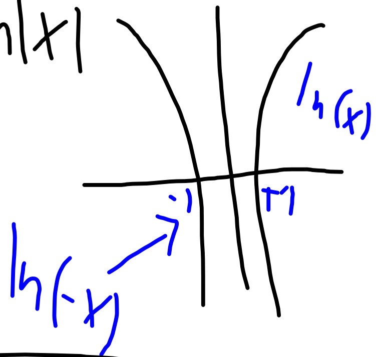
$$\therefore \boxed{\frac{d}{dx}(\sin x) = \cos x}$$

Om $\sin\left(x + \frac{\pi}{2}\right) = \cos x$

$$\frac{d}{dx}(\cos x) = \frac{d}{dx} \sin\left(x + \pi/2\right) = \cos\left(x + \pi/2\right) = -\sin x$$

$$\therefore \frac{d}{dx}(\cos x) = -\sin x$$

Derivator av Elementära fkner

$f(x)$	$f'(x) = \frac{df}{dx}$
$k =$ konstant	0
$\alpha \in \mathbb{R}$, x^{α}	$= \alpha x^{\alpha-1}$, där x är definierad
e^x	e^x
$\ln\|x\|$ 	$\frac{1}{x}$, $x \neq 0$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x = \frac{\sin x}{\cos x}$	$1 + \tan^2 x = \frac{1}{\cos^2 x}$
$\cot x = \frac{\cos x}{\sin x}$	$-(1 + \cot^2 x) = -\frac{1}{\sin^2 x}$

Ex Finn ekv för tangenten och normalen till $f(x) =$

Deriverings regler

1. $\left(A f(x) + B g(x)\right)' = A f'(x) + B g'(x)$

   där A, B konstanter

2. $\left(f(x) g(x)\right)' = f'(x) g(x) + f(x) g'(x)$

3. $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x) g(x) - f(x) g'(x)}{\left(g(x)\right)^2}$

Spec om $f(x) = 1 \Rightarrow f'(x) = 0$

$$\left(\frac{1}{g(x)}\right)' = -\frac{g'(x)}{\left(g(x)\right)^2}$$

Ex

$$\frac{d}{dx}\left(\frac{1}{\ln|x|}\right) = \left[\begin{array}{l} g(x) = \ln|x| \\ g'(x) = \frac{1}{x} \end{array}\right]$$

$$= \frac{-\frac{1}{x}}{\left(\ln|x|\right)^2} = -\frac{1}{x\left(\ln|x|\right)^2}$$

Ex Finn ekvationer tangenten och normalen till kurvan

$$f(x) = \frac{x+1}{x-1} \quad i \ x = 2$$

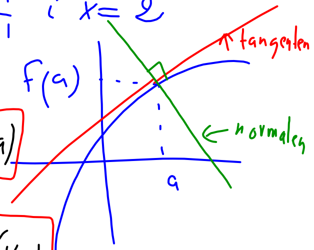
Lösning

tangenten ges

$$\boxed{y - f(a) = f'(a)(x-a)}$$

normalen

$$\boxed{y - f(a) = -\frac{1}{f'(a)}(x-a)}$$



$x = 2 \Rightarrow f(2) = \frac{2+1}{2-1} = 3$

Sök $f'(2)$

$$f(x) = \frac{x+1}{x-1} = \frac{u}{v} \Rightarrow f'(x) = \frac{u'v - uv'}{v^2}$$

$$\therefore f'(x) = \frac{\frac{d}{dx}(x+1)(x-1) - (x+1)\frac{d}{dx}(x-1)}{(x-1)^2}$$

$$= \frac{1(x-1) - (x+1)1}{(x-1)^2}$$

$$= \frac{\cancel{x} - 1 - \cancel{x} - 1}{(x-1)^2} = -\frac{2}{(x-1)^2}$$

$$\therefore f'(2) = -\frac{2}{(2-1)^2} = -2 \ \#$$

$$y - f(a) = f'(a)(x-a) \Leftrightarrow y - 3 = -2(x-2)$$

$$\Leftrightarrow y + 2x = 7$$

$$y - f(a) = -\frac{1}{f'(a)}(x-a) \Leftrightarrow y - 3 = -\frac{1}{(-2)}(x-2)$$

$$\Leftrightarrow 2y - x = 4$$

SVAR tangenten: $y + 2x - 7 = 0$

Normalen $2y - x - 4 = 0$